1 Introduction

In this paper four-wheel steering (4WS) vehicle handling configurations are developed using combination of feedforward and feedback controllers. These configurations allow robust disturbance rejection, robust sideslip angle attenuation and robust model matching against parameters variations and rejection of lateral forces torque disturbances which may rise from wind forces. Control synthesis is conducted on a linear fractional transformation (LFT) model in order to represent vehicle road adhesion variations [6] while simulation tests are conducted on a nonlinear model.

The paper is organized as follows: section 2 introduces the car model which is used for active steering controller analysis. Vehicle LFT model, controller objectives and synthesis methodology are presented in section 3. Controller implementation and several simulation results are given in section 4.

2 Lateral motion modeling

A simple nonlinear model of a vehicle is derived by neglecting the roll and pitch motions. This model includes the two translational motions and the yaw motion. According to Figure 1, the vehicle wheels are numbered from 1 to 4. The interaction between the tire and road surface is decomposed into longitudinal forces \( f_{x_i} \) and lateral forces \( f_{y_i} \). Here the longitudinal forces are only computed such that the vehicle forward speed remains near the desired value while the lateral forces are computed from the tire slip angles \( f_{y_i}(\alpha_i) \).

The nonlinear model is obtained by writing the translational and rotational equations in the vehicle fixed frame

\[
\begin{bmatrix}
    m \left( \dot{x}_G - j_G \dot{\alpha} \right) \\
    m \left( \dot{y}_G + \dot{x}_G \dot{\alpha} \right)
\end{bmatrix}
= \begin{bmatrix}
    R_1 & R_2
\end{bmatrix}
\begin{bmatrix}
    f_{x_f} \\
    f_{y_f}
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    \frac{1}{l_w}
\end{bmatrix}
\begin{bmatrix}
    f_x \\
    f_y
\end{bmatrix}
\]

(1)

where

\[
R_1 \left( \delta_f \right) = \begin{bmatrix}
    \cos \delta_f & -\sin \delta_f \\
    \sin \delta_f & \cos \delta_f
\end{bmatrix},
\]

\[
R_2 \left( \delta_r \right) = \begin{bmatrix}
    \cos \delta_r & -\sin \delta_r \\
    \sin \delta_r & \cos \delta_r
\end{bmatrix}
\]

and

\[
\begin{aligned}
    f_{x_f} &= f_{x_1} + f_{x_2} \\
    f_{y_f} &= f_{y_1} + f_{y_2} \\
    f_{x_r} &= f_{y_3} + f_{y_4} \\
    f_{y_r} &= f_{y_3} + f_{y_4}
\end{aligned}
\]

The distances, the longitudinal and lateral forces are shown on Figure 1. Vehicle parameters are given in the appendix.

The model used for control synthesis is derived from the previous model in which the longitudinal velocity is assumed to be constant \( \dot{x}_G = v, \dot{x}_G = 0 \) and all the angles are assumed to be small [1]. One can then use the sideslip angle
\[
\sin \beta = \frac{\Delta \omega}{v} \approx \beta \]

as a state variable. It is also assumed equal cornering stiffnesses for the two front wheels (\( \frac{\Delta \omega}{c_1} = c_1 = c_2 \)) and the rear ones (\( \frac{\Delta \omega}{c_3} = c_3 = c_4 \)). When the wheel base is neglected, the tire slip angles are reduced to (\( \alpha_1 = \alpha_2 = \alpha_f \)) and (\( \alpha_3 = \alpha_4 = \alpha_r \)). These angles are also considered small and thus the equations (\( f_f(\alpha_f) = c_1 \alpha_f \)) and (\( f_r(\alpha_r) = c_3 \alpha_r \)) are used and leads to the well known single-track model.

For control synthesis, the longitudinal speed is set to \( 20 m/s \) and all tire cornering stiffnesses are at their nominal values.

<table>
<thead>
<tr>
<th>Tire slip angles</th>
</tr>
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<tbody>
<tr>
<td>( \alpha_1 = \delta_f - \tan^{-1}\left( \frac{\sin \beta - (v \cos \delta_f) \delta_f - (v \cos \delta_r) \delta_r}{v \cos \beta + (v \sin \delta_f) \delta_f + (v \sin \delta_r - \frac{\Delta \omega}{v}) \delta_r} \right) )</td>
</tr>
<tr>
<td>( \alpha_1 \approx \delta_f - \left( \beta + \frac{\Delta \omega}{v} \right) )</td>
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<tr>
<td>( \alpha_2 = \delta_f - \tan^{-1}\left( \frac{\sin \beta - (v \cos \delta_f) \delta_f - (v \cos \delta_r) \delta_r}{v \cos \beta + (v \sin \delta_f) \delta_f + (v \sin \delta_r + \frac{\Delta \omega}{v}) \delta_r} \right) )</td>
</tr>
<tr>
<td>( \alpha_2 \approx \delta_f - \left( \beta + \frac{\Delta \omega}{v} \right) )</td>
</tr>
<tr>
<td>( \alpha_3 = \delta_r - \tan^{-1}\left( \frac{\sin \beta - (v \cos \delta_f) \delta_f - (v \cos \delta_r) \delta_r}{v \cos \beta + (v \sin \delta_f) \delta_f + (v \sin \delta_r - \frac{\Delta \omega}{v}) \delta_r} \right) )</td>
</tr>
<tr>
<td>( \alpha_3 \approx \delta_r - \left( \beta - \frac{\Delta \omega}{v} \right) )</td>
</tr>
<tr>
<td>( \alpha_4 = \delta_r - \tan^{-1}\left( \frac{\sin \beta - (v \cos \delta_f) \delta_f - (v \cos \delta_r) \delta_r}{v \cos \beta + (v \sin \delta_f) \delta_f + (v \sin \delta_r + \frac{\Delta \omega}{v}) \delta_r} \right) )</td>
</tr>
<tr>
<td>( \alpha_4 \approx \delta_r - \left( \beta - \frac{\Delta \omega}{v} \right) )</td>
</tr>
</tbody>
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Table 1. Tire slip angles

### 3 Design methodology

The input \( \delta_f \) is set by the driver while the steering angle \( \delta_r \) of the rear wheels is made up of a feedback part from the yaw rate \( (y = \psi = r) \) and a feedforward part computed from the driver assigned front steering angle.

\[
\delta_r(s) = K_{fb}(s) r(s) + K_{ff}(s) \delta_f(s) \tag{2}
\]

where \( K_{fb} \) is the feedback controller and \( K_{ff} \) is the feedforward controller. It must be noted that the feedforward and the feedback controller components may be also used individually. This leads to 3 different rear wheels steering configuration. Concerning the feedforward controller, two design procedure are developed here. In the first one, the feedforward controller is designed such that the sideslip angle is minimized. In the second one it is designed in order that the transfer function from \( \delta_f \) to the yaw rate \( r \) follows a given ideal reference model \( T_0 \).

#### 3.1 Feedback control synthesis

The purpose of the feedback controller is two folds: The first is the rejection of the perturbations induced by the external inputs such as wind forces, the second is to improve the damping of the yaw motion especially at high speed and low road adhesion.

The feedback controller \( K_2 \) is synthesized using the non iterative method of [7] (Figure 2). It is well known that in this case, the maximum stability margin depends only on the system not on the controller. It is enhanced by shaping the open loop transfer function \( P_{yu} \) from the rear steering angle \( u = \delta_r \) to the measured yaw rate \( (y = r) \). In order to reject a constant step input perturbation on the yaw rate, a weighting compensator of the form of a high DC gain lag filter is introduced [7].

\[
W(s) = \frac{1000}{0.1s + 1} \frac{100s + 1}{100s + 1} \tag{3}
\]

This leads to a shaped plant \( P_s \) given by [7]

\[
P_s = P_{yu}W \tag{4}
\]

The maximum stability margin \( \varepsilon_{max} \) is directly computed from equation 5

\[
\varepsilon_{max} = \left\{ 1 - \left[ \bar{M}_s \bar{N}_s \right]_H^2 \right\}^{1/2} \tag{5}
\]

\[
= \left( \inf_{K_{stab}} \left\{ \frac{1}{1 - P_sK_2} \left[ \begin{array}{c} K_2P_s I \\ P_s K_2 \end{array} \right] \right\} \right)^{-1} \tag{6}
\]

where \( (\bar{M}_s, \bar{N}_s) \) are the normalized left coprime factors of the shaped plant. In the following, a sub optimal controller \( K_2 \) is computed with a stability margin of \( (\varepsilon_p = 0.9 \varepsilon_{max}) \). The implemented feedback controller \( K_{fb} \) is obtained by combining \( K_2 \) with the weighting filter: \( K_{fb} = W K_2 \) (Figure 2).

#### 3.2 Robust feedback controller synthesis

In the following, it is assumed that the wind forces are handled by the feedback part of the controller, so they are not further considered. The state vector is the one of the single-track model \( x = [\beta \ \psi]^T \).

First of all, this single-track is used in order to derive an LFT model of the vehicle where parameters variations are modeled as a diagonal perturbation matrix \( \Delta \). Thus a feedback controller is proposed, which ensures a guaranteed \( H_{\infty} \) performance index for the whole LFT models.

#### 3.2.1 Vehicle LFT modeling

In order to take into account road adhesion degradation, variations of the cornering stiffnesses are assumed to have the following forms, derived from the nominal values \( c_{f0} \) and \( c_{r0} \) [6]

\[
\begin{align*}
&c_f = c_{f0}(1 + \sigma_f \delta_f), \quad ||\delta_f|| \leq 1 \\
&c_r = c_{r0}(1 + \sigma_r \delta_r), \quad ||\delta_r|| \leq 1 
\end{align*} \tag{7}
\]

where \( \sigma_f, \sigma_r \) are scaling factors used to reflect the magnitude of the perturbation.

In order to transform the parameter varying single-track model in the form of an LFT model, 2 fictitious inputs and outputs are needed.

- one fictitious input and output for the variation of the front cornering stiffness

\[
p_1 = \delta_f q_1 \quad q_1 = -\beta + \delta_f - \frac{\Delta \omega}{c_{r0}} \tag{8}
\]
The fictitious input and output vectors are related by
\[
p = \Delta q
\]
where the diagonal perturbation matrix has the following structure
\[
\Delta = \text{diag}\{\delta_1, \delta_2\}
\]
(10)

It has to be noted that the bicycle model may be recovered by setting \(\sigma_f = \sigma_r = 0\). A regulated output \(z\) is introduced and will be chosen according to the performance index.

Finally the corresponding transfer matrix for system (8) is
\[
P = \begin{bmatrix}
P_{qp} & P_{q\delta_f} & P_{q\delta_r} \\
P_{zp} & P_{z\delta_f} & P_{z\delta_r} \\
P_{zp} & P_{z\delta_i} & P_{z\delta_r}
\end{bmatrix}
\]

(11)

Considering vehicle rear wheel active steering, Equation (2) takes the form
\[
\delta_r = \delta_{ff} + \delta_{fb}
\]
where \(\delta_{fb} = K_{fb}\delta_f\) is the feedback component which is computed at the first stage using the loop shaping coprime synthesis, and \(\delta_{ff}\) is the additional feedforward steering angle (Figures 2 and 3).

The system \(G\) including the feedback controller is obtained by lower linear fractional transformation (\(\mathcal{F}_1\)) and can be written as
\[
\begin{bmatrix}
q \\
z
\end{bmatrix} = \begin{bmatrix}
G_{qp} & G_{q\delta_f} & G_{q\delta_f} \\
G_{zp} & G_{z\delta_f} & G_{z\delta_f}
\end{bmatrix}\begin{bmatrix}
p \\
\delta_f \\
\delta_{ff}
\end{bmatrix}
\]

(13)

3.2.2 Feedforward for sideslip angle minimization

Here the closed loop controller is assumed to be used, however one can decide to only use the feedforward part. In equation (14), \(K_{fb}\) is set to zero. The output, \(z = \beta\) is chosen as the regulated output. In order to include some frequency domain considerations, this output is weighted by a \(W_0\). The new regulated output \(z_0\) is given by \(z_0 = W_0z\), and system of equation (14) is modified to
\[
G_{ff} = \begin{bmatrix}
I & 0 \\
0 & W_0
\end{bmatrix}G
\]

(15)

The feedforward part of the controller \(K_{ff}\) is synthesized with the objective to keep the weighted signal small for an entire family of perturbed plants described by the previous upper LFT model. The transfer function from the front steering angle to the weighted output \(z_0\) is obtained from (16) (Figure 2).

\[
T_{z\delta_f} = \mathcal{F}_u\left(G_{ff}\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}, \Delta\right)
\]

(16)

3.2.3 Feedforward for model reference matching

Let now \(T_0\) be the desired transfer function between \(\delta_f\) and the yaw rate \(r\). The difference between the actual and the desired yaw rate is also weighted by a filter \(W_0\). The weighted
error is given by
\[ z_0 = W_0 (r - T_0 \delta_f) \]  
(17)

The reference model \( T_0 \) is chosen as first order transfer function with the same steady state gain as the conventional car of the form
\[ T_0 = \frac{G(t_0)}{0.2s + 1} \]  
(18)

This model will ensure the same steady state value for the controlled and the conventional car. The settling time is about 0.8 s. The reference model is chosen as a first order in order to avoid any overshoot on vehicle responses.

Equations (14) and (16) are thus updated to (Figure 3)
\[ G_{ff} = \begin{bmatrix} I & 0 \\ 0 & W_0 \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & T_0 \end{bmatrix} \]  
(19)

The feedforward controller has to keep the error signal \( z_0 \) small for the entire family of perturbed plants described above. The transfer function from the disturbance \( \delta_f \) to \( z_0 \) takes the same form as (16), only matrix \( G_{ff} \) are different.

In order to favor the medium frequencies, in the two cases, the weighting filter is chosen as
\[ W_0 = 0.01 \frac{10s + 1}{s + 1} \]  
(20)

3.2.4 \( H_\infty \) robust performance index

For the two cases, we define the \( H_\infty \) robust performance level \( \gamma_f \) which has to be ensured
\[ \sup_{\Delta \in \mathcal{H}_\Delta} \| T_{z_0 \delta_f} \|_{L_2 \to L_2} < \gamma_f \]  
(21)

where the set \( \mathcal{H}_\Delta \) is defined in [8].

According to equation 14, it is assumed in the following that the state-space realization of \( G_{ff} \) is given by
\[ G_{ff}(s) = \begin{bmatrix} A & B & E & M \\ C_1 & L_1 & H_1 & N_1 \\ C_2 & L_2 & H_2 & N_2 \end{bmatrix} \]  
(22)

A necessary and sufficient condition for the existence of a controller satisfying the performance index (21) is given by the 3 LMIs below [8]
\[ \bar{\mathcal{M}}_R < 0 \]

where \( \bar{\mathcal{M}}_R = \text{diag} \{ \bar{\mathcal{N}}_R, I \} \), \( \bar{\mathcal{N}}_R \) is a matrix whose columns constitute a basis of the kernel of \( \begin{bmatrix} M^T & N_1^T & N_2^T \end{bmatrix} \), \( R \) and \( Z \) are symmetric definite matrices and \( X \) is a diagonal matrix with the same structure as \( \Delta \). The controller is then reconstructed using classical \( H_\infty \) optimization [8] [2]. The process is the following

- form the matrices \( Q = \begin{bmatrix} X^{-1} & 0 \\ 0 & I \end{bmatrix} \) and
\[ Q_{\gamma_f} = \begin{bmatrix} X & 0 \\ 0 & \gamma_f^{-1}I \end{bmatrix} \]
- compute the state space form of the system \( G_Q \) given by
\[ G_Q(s) = \begin{bmatrix} A & BQ_{\gamma_f} & E \\ Q_1 & QL_1 & QH_1Q_{\gamma_f} \\ 0 & Q_1 & QN_1 \end{bmatrix} \]
(23)

- finally perform the \( H_\infty \) standard optimization (25) to get the feedforward controller \( K_{ff} \)
\[ \| F_{\delta f} (G_Q, K_{ff}) \|_\infty \leq 1 \]  
(25)

4 Simulation results

For testing effect of the various developed controllers, two vehicle maneuvers are simulated: wind force rejection and lane change. All simulations are conducted on the nonlinear model and two different speed and road adhesion, dry and wet, are considered. In addition for comparing design results, the following three control configurations are considered

- Conventional car: in this case, the front wheel may be steered by the driver while the rear ones can not be steered. The vehicle has thus no reaction to wind force during driver reaction time. Solid line style is used in all the figures for time response plots.
- 4WS car with front wheel steering angle feedforward: the rear wheels are automatically steered using the information about the driver front wheel steering angle, through
the feedforward controller. The purpose here is sideslip angle minimization. Dashed line style is used in all the figures for time response plots.

- **4WS car with yaw rate feedback and front wheel steering angle feedforward**: this case corresponds to the fully controlled rear wheels for better damping and robust model matching or sideslip minimization. Dash-dot line style is used.

It must be noted that robustness face to speed variations is considered but controllers can also be speed scheduled with stability preserving as detailed in [5]. In such a case, the controller implementation needs three type of measurement: measurement of the yaw rate by a gyro, the speed by an odometer and the steering wheel angle by an encoder. In addition the transfer functions of the two controller component are discretized using bilinear transformation.

### 4.1 Disturbance rejection

Firstly a step disturbance wind force is applied to the vehicle at nominal speed and full road adhesion. The wind force appears at time \( t_1 = 1s \) and disappears at \( t_2 = 2s \). It is assumed that the driver doesn’t react to this disturbance \( (\delta_f = 0) \). One can note from Figure 5 that the yaw rate steady state value is drastically reduced within driver reaction time. This mean that the controlled vehicle will stay closer to the assigned trajectory. On other hand, the maximum value of yaw rate during the transient phase is smaller than the one of the conventional car. Rear wheel steering angle is limited to 3.1 deg. One can notice however, that the controlled car presents greater sideslip angle.

Responses for \( v = 40m.s^{-1} \) and adhesion reduced to the half are given in Figure 6. The controller exhibits good stability and performance robustness, in fact comparing to the conventional car, responses are still better damped. The disturbance is rejected and sideslip is not much increased.

### 4.2 Lane change maneuver

The handling improvement is now investigated in case of driver steering angle which corresponds to lane change maneuver. The dashed line corresponds to the configuration with only the feedforward part and the dash-dot line corresponds to the response of the configuration with feedback and feedforward. In both cases, the feedforward is designed for sideslip angle minimization. Figure 7 shows results obtained at nominal speed with common road adhesion equal to 1. Figure 8 shows results obtained for \( v = 40m.s^{-1} \) and road adhesion reduced to the half. One can see from the figures that the controlled car sideslip angle is reduced for both controlled cars, the responses are well damped when feedback is used, especially those of the yaw rate even for the perturbed case.

Additional Results for model reference matching are given in [3].
5 Conclusion

In this paper, combination of feedforward and feedback $H_\infty$ controllers applied to active steering for vehicle handling improvement of 4WS vehicles has been presented. The synthesis methodology simply allows direct specification of several design objectives. Controller performances are assessed on two typical maneuvers where the controlled vehicle which combines feedforward and feedback controllers is found to exhibit better performance and stability properties. In addition the chosen control structure can be made speed scheduled easily [5].

References


Appendix

- $c_f$: front cornering stiffness (57566 N.rad$^{-1}$)
- $c_r$: rear cornering stiffness (57512 N.rad$^{-1}$)
- $\delta_f$: front wheels steering angle
- $\delta_r$: rear wheels steering angle
- $I_z$: yaw moment of inertia (1574 kg.m$^2$)
- $l_f$: distance CG to front axle (1.065 m)
- $l_f' = l_f - n_t$
- $l_r$: distance CG to rear axle (1.462 m)
- $m$: total mass (991 kg)
- $n_t$: tire length contact (0.113 m)
- $v$: longitudinal velocity ([3, 51] m.s$^{-1}$)