Actuator Fault diagnosis: H_∞ framework with relative degree notion

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Abstract: A new actuator fault diagnosis and estimation approach is proposed for dynamical systems. The main contribution consists in enhancing the fault detection with a new observer that takes into account the relative degree of the output of the system with respect to the fault. The Single Input Single Fault (SIFO) case is considered to present the approach and an extension to systems with multiple outputs and multiple faults. The convergence of the proposed residual generator is analyzed using the Lyapunov theory which can be expressed straightforwardly in terms of Linear Matrix Inequalities (LMIs). Numerical examples are provided in order to illustrate the proposed approach.

1. INTRODUCTION

These last years, the problem of fault diagnosis in dynamical systems has taken an important place in engineering control. This is due to the great demand in high performance control systems even in the presence of faults. For example, in industrial production and transportation fields, the systems should operate without time discontinuity in order to meet demand and to avoid delays that may cause considerable loss of money. Furthermore, systems must be designed to ensure the safety of the human operator and system itself such as in aeronautics and aerospace vehicles. The components of the system (actuators, sensors,...) may be affected by a fault which can cause some unpleasant repercussions on the system and the human operator. Therefore, the control systems should be designed by taking into account the faulty scenarios. Some results have been reported in the context of fault diagnosis (fault detection, isolation and estimation) using different approaches: namely, signal processing, model-based, soft computing approaches,..., (Patton et al. [2001],Gertler [1998],Frank and Ding [1997]). They aim to monitor the system and provide some information to be used in the control task to compensate the faults and preserve the healthy system functioning with adequate fault tolerant controllers.

In the context of model-based fault diagnosis, the H_∞ approach is one of the most interesting techniques in designing residual generators that provide fault indicators and allow to detect, isolate and estimate the magnitude of the faults. The presence of disturbances and measurement noises may affect the problem of fault detection and isolation, by masking the effect of the fault, which causes a delay in the detection of the fault and hence may lead to disaster consequences. Therefore, the H_∞ framework has been extended by using the H_- performance index in order to enhance the sensitivity of the fault on the residual signal. Several interesting results are reported in the literature. In Chadli et al. [2013], the H_-/H_∞ has been exploited in order to generate residual signals that are made sensitive as possible for the faults and less sensitive as possible to the disturbances and the measurement noises. In Ichalal et al. [2014], the min/max problem is transformed into a simple minimization problem by introducing a filter that aims to enhance the sensitivity of the residual signal to the fault and minimizes the effect of the disturbances. In Mazars et al. [2008], a reference model is used in order to shape the response of the residual signal. An adequate choice of the reference model can significantly enhance the sensitivity of the residual signal to the fault.

After this brief bibliography, it has been noted that, in general, the same faults affecting the state equation affect also the output equation. This commonly used assumption allows to guarantee the regularity assumption needed in the H_∞ framework, which lead to good performances of the residual generators. However, in real systems, actuator faults are different of the sensor ones, then, the regularity assumption is not satisfied, hence, degraded performances of the residual generator are obtained which affect the fault sensitivity. A solution to this issue is given in Ichalal et al. [2014] by perturbing the output of the system with the actuator faults and a small parameter. Acceptable performances are then obtained regarding to the fault sensitivity, but these performances depend on the fixed small parameter.

In this paper, a new solution for the problem of actuator fault detection and isolation is proposed by using the relative degree notion. This work is a continuation of the result given in Ichalal et al. [2014]. The use of relative degree notion aims to define new auxiliary outputs depending on the actuator faults. Indeed, by differentiating the outputs of the system according to the relative degrees with respect to the faults, new outputs can be generated and the system with the new output vector satisfies the regularity condition. Of course, the implementation of the proposed approach is based on time derivatives of the noisy outputs which are obtained by the recent robust algorithms that provides high order time derivatives with good convergence properties and insensitivity to measurement noises. For example, one can cite the High Order Sliding Mode Differentiator (HOSMD) having a finite time convergence property Levant [2003], the Non-Asymptotic algebraic differentiator in Fliess et al. [2008], the Linear Time Varying differentiator in Ibrir [2003] and the High gain differentiator in Kalsi et al. [2010]. In this work, the HOSMD is used.

Throughout the paper, the following definitions and lemmas will be used.

Definition 1. Relative Degree. Isidori [1995] Consider the linear system

$$\dot{x}(t) = Ax(t) + Ef(t), \quad y(t) = Cx(t) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $f \in \mathbb{R}$ is the fault signal and $y \in \mathbb{R}$ is the output signal. The relative degree of the system (1) is the number r satisfying

$$\begin{cases} CA^{i-1}E = 0, \forall i = 1, ..., (r-1) \\ CA^{r-1}E \neq 0 \end{cases} \tag{2}$$

In other words, the relative degree corresponds to the number of successive time derivatives of the output to obtain an equation that involves the fault signal f

$$y^{(r)}(t) = CA^r x(t) + \underbrace{CA^{r-1}E}_{\neq 0} f(t) \tag{3}$$

Definition 2. Bounded Real Lemma. Boyd et al. [1994] For the system

$$\dot{x}(t) = Ax(t) + Ef(t), \; y(t) = Cx(t) + Df(t) \tag{4}$$

The system (4) is stable and satisfy the conditions

$$\begin{cases} \lim\limits_{t \to +\infty} y = 0 \quad \text{if } f = 0 \\ \|y(t)\|_2 < \gamma \|f\|_2 \text{ if } f \neq 0 \end{cases} \tag{5}$$

if the following LMI is satisfied

$$\begin{pmatrix} A^T P + PA & PE & C^T \\ E^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix} < 0 \tag{6}$$

with P, a symmetric and positive definite matrix. Furthermore, if $\gamma > 0$ is minimized, the transfer from f to y is also minimized.

2. PROBLEM STATEMENT AND MOTIVATION

Let us consider the linear system subject to an actuator fault

$$\begin{cases} \dot{x}(t) = Ax(t) + Ef(t) \\ y(t) = Cx(t) \end{cases} \tag{7}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f(t) \in \mathbb{R}$ is the actuator fault and $y(t) \in \mathbb{R}$ is the system output. The matrices A, E and C are real constant with appropriate dimensions.

Note that without loss of generality, the control input of the system is omitted (i.e. $u(t) = 0$), the extension to systems having $u(t) \neq 0$ is straightforward. Assume that the output $y(t)$ has a relative degree r with respect to the fault $f(t)$ and the system is observable (i.e. the pair (C, A) is observable).

Classically, an H_∞ Residual Generator is described by the following equations

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + L(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C\hat{x}(t) \\ r(t) = M(y(t) - \hat{y}(t)) \end{cases} \tag{8}$$

without taking into account the relative degree of the system. The matrices L and M are real constant to be determined (For the Single Fault and Single Output system, M is reduced to a scalar variable). In order to make the residual $r(t)$ sensitive to the fault $f(t)$, a virtual residual $r_e(t)$ can be defined by the equation

$$r_e(t) = r(t) - f(t) \tag{9}$$

Let us define the state estimation error $e(t) = x(t) - \hat{x}(t)$. The system generating the virtual residual signal $r_e(t)$ is then described as follows

$$\begin{cases} \dot{e}(t) = (A - LC)\, e(t) + Ef(t) \\ r_e(t) = MCe(t) - f(t) \end{cases} \tag{10}$$

Consequently, the gain matrices L and M should be designed in such a way to minimize the effect of $f(t)$ with respect to $r_e(t)$. It is then clear that if $r_e(t) \to 0$, the real residual signal $r(t)$ tends to the fault $f(t)$ i.e. $r(t) \to f(t)$ which allows to detect the fault $f(t)$ (note that if $r_e(t) = 0$, we have $r(t) = f(t)$ which provides the fault estimation, this is the ideal case which cannot be achieved by the H_∞ approach).

In standard H_∞ framework, the matrices L and M of the system (10) should be determined in such a way to satisfy the following constraints

$$\begin{cases} \lim\limits_{t \to +\infty} r_e(t) = 0 \quad \text{if } f(t) = 0 \\ \|r_e(t)\|_2 < \gamma \|f(t)\|_2 \text{ if } f(t) \neq 0 \end{cases} \tag{11}$$

In the presence of the fault $f(t)$, the sensitivity of $r(t)$ with respect to $f(t)$ is better when the positive real γ is as small as possible. By using the Bounded Real Lemma, the problem of determining the matrices L and M is expressed as an optimization problem given by

$$\min_{P,K,M} \bar{\gamma}$$

s.t.

$$\begin{pmatrix} A^T P + PA - C^T K^T - KC & PE & C^T M^T \\ E^T P & -\gamma & -1 \\ MC & -1 & -\gamma \end{pmatrix} < 0 \tag{12}$$

where $P = P^T > 0$. After solving the optimization problem, the matrices of the residual generator are obtained by $L = P^{-1}K$ and M is obtained directly. The attenuation level is given by γ.

By analyzing the LMI (12), it appears that under the observability condition, the LMI can admit a solution. Then, if the LMI is negative definite, we have

$$\begin{pmatrix} -\gamma & -1 \\ -1 & -\gamma \end{pmatrix} < 0 \tag{13}$$

which is equivalent, by Schur Complement, to $\gamma > 1$. Then the best value for γ is $1 + \epsilon$ where ϵ is a positive small number. This illustrates clearly the limitation of such an approach.

In this paper the H_∞ approach is enhanced by exploiting an intrinsic notion, of the system, which is the relative degree. The motivation of the proposed approach is the fact, that recently, interesting and efficient approaches were proposed to estimated the high order time derivatives of a signal. Namely, the approaches based on sliding mode theory which provide an estimate of a finite order time derivatives of a given signal in finite time Levant [2003]. Another interesting approach is the numerical differentiator proposed in Fliess et al. [2008] which provide also a good non-asymptotic estimation of the time derivatives. Finally, the third interesting approach is the asymptotic one given in Ibrir [2003] which is based on high gain notion and Linear Time Varying differentiator. The time derivatives obtained by these approaches are better than the classical differentiators which are very sensitive to the noises affecting the signal.

3. H_∞ RESIDUAL GENERATOR WITH RELATIVE DEGREE CONSIDERATION

In this section, a simple approach is proposed in order to enhance the performances of the H_∞ Residual Generator and overcome the problem of the the attenuation level γ greater than 1. Let us consider the system (7), where the relative degree is r. This means that

$$y^{(r)}(t) = CA^r x(t) + CA^{r-1}Ef(t) \tag{14}$$

Now, let us consider the new output $\tilde{y}(t)$ defined by

$$\tilde{y}(t) = \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(r)}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} C \\ CA \\ \vdots \\ CA^r \end{pmatrix}}_{\tilde{C}} x(t) + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ CA^{r-1}E \end{pmatrix}}_{R} f(t) \tag{15}$$

The system with the new generated output becomes

$$\begin{cases} \dot{x}(t) = Ax(t) + Ef(t) \\ \tilde{y}(t) = \tilde{C}x(t) + Rf(t) \end{cases} \tag{16}$$

The proposed residual generator is

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + L(\tilde{y}(t) - \hat{\tilde{y}}(t)) \\ \hat{\tilde{y}}(t) = \tilde{C}\hat{x}(t) \\ r(t) = M(\tilde{y}(t) - \hat{\tilde{y}}(t)) \end{cases} \tag{17}$$

By defining the state estimation error $e(t) = x(t) - \hat{x}(t)$ and the virtual residual signal $r_e(t) = r(t) - f(t)$, it follows

$$\begin{cases} \dot{e}(t) = \left(A - L\tilde{C}\right) e(t) + (E - LR)f(t) \\ r_e(t) = M\tilde{C}e(t) + (MR - 1)\, f(t) \end{cases} \tag{18}$$

The objective is then to determine the matrices L and M which ensure the conditions (11) with the minimization of attenuation level γ for maximal sensitivity of $r(t)$ with respect to $f(t)$. The following theorem provides an optimization problem under LMI conditions aiming the determination of L and M with minimal transfer from $f(t)$ to $r_e(t)$ corresponding to a maximal transfer from $f(t)$ to $r(t)$.

Theorem 1. Under the observability of the pair (C, A) and the relative degree r $(1 \leq r \leq n)$, the residual generator exists if there exist a symmetric and positive definite matrix P, gain matrices K and M and a positive scalar γ solution to the following optimization problem

$$\min_{P,K,M} \gamma$$

s.t.

$$\begin{pmatrix} A^TP + PA - \tilde{C}^TK^T - K\tilde{C} & PE - K\tilde{C} & \tilde{C}^TM^T \\ E^TP - \tilde{C}^TK^T & -\gamma & R^TM^T - 1 \\ M\tilde{C} & MR - 1 & -\gamma \end{pmatrix} < 0 \tag{19}$$

The gain L of the residual generator is obtained from the equation $L = P^{-1}K$. The attenuation level γ describes the sensitivity of $r(t)$ with respect to $f(t)$. The smallest is γ the greatest is the sensitivity.

Proof. The proof of the theorem 1 is straightforward. It is based on the Bounded Real Lemma applied to the system (18). This allows to obtain the following inequality

$$\begin{pmatrix} A^TP + PA - \tilde{C}^TL^TP - PL\tilde{C} & PE - \tilde{C}^TL^TP & \tilde{C}^TM^T \\ E^TP - PL\tilde{C} & -\gamma & R^TM^T - 1 \\ M\tilde{C} & MR - 1 & -\gamma \end{pmatrix} < 0 \tag{20}$$

Finally, by considering the change of variable $K = PL$ and the objective of minimizing the transfer from $f(t)$ to $r_e(t)$, the optimization problem given in the theorem 1 is obtained.

The negativity of (19) implies that the following inequality holds

$$\begin{pmatrix} -\gamma & R^TM^T - 1 \\ MR - 1 & -\gamma \end{pmatrix} < 0 \tag{21}$$

which is equivalent to

$$\gamma^2 > \left(R^TM^T - 1\right)(MR - 1) \tag{22}$$

Since this paper considers only systems with single fault and single output, the term $(MR - 1)$ is just a scalar, it follows

$$\gamma^2 > (MR - 1)^2 \tag{23}$$

Taking the square root, on obtains

$$\gamma > MR - 1 \tag{24}$$

Since $R \neq 0$ due to the relative degree, it is then possible to chose M in such a way to have $(MR - 1) \to 0$. Thus, the parameter $\gamma > 0$ may takes values small than 1 which enhance the residual sensitivity with respect to the fault compared to the classical approach where $\gamma > 1$.

4. ILLUSTRATIVE EXAMPLE

To illustrate the performances of the proposed residual generator design approach, let us consider the system (7) with the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, E = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{25}$$

The system is observable and the output $y(t)$ have a relative degree 2 with respect to the fault $f(t)$.

4.1 Classical approach

For the classical approach, solving the optimization problem under the LMI constraint (12) leads to the following solution

$$P = 10^4 \times \begin{pmatrix} 2.082 & -0.0009 \\ -0.0009 & 0.0000 \end{pmatrix},$$

$$L = 10^6 \times \begin{pmatrix} 0.0009 \\ 1.9286 \end{pmatrix}, \; M = -9.2522$$

The obtained attenuation level is $\gamma = 1.001$ which is greater than 1 as expected (see section 2). The figure 1 illustrates the fault and the associated residual obtained by the classical H_∞ residual generator. It can be seen that the residual detects the fault but with a very small magnitude (around 5×10^{-6}). This leads, in noisy measurements, to a non-detection of the fault because the residual risks to be masked by the measurement noises.

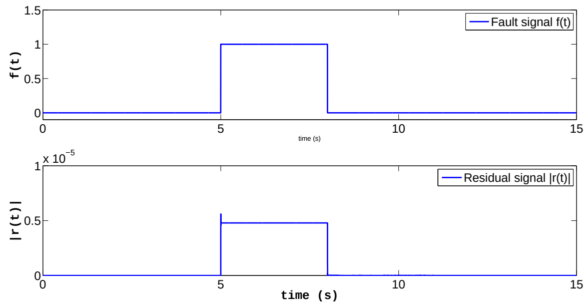

Fig. 1. Fault and residual signal (classical approach)

4.2 Proposed approach

For the proposed approach, first, the relative degree is computed and it is given by $r = 2$. From this information, the design approach needs the knowledge of both the first and the second time derivatives of the output $y(t)$. These derivatives are obtained by the third order sliding mode differentiator (Levant [2003]). After computing the matrices $\tilde{C}$ and R as in equation (15), the optimization problem given in the theorem 1 provides the following results

$$P = \begin{pmatrix} 1.1417 & 0 \\ 0 & 1.1417 \end{pmatrix},$$

$$L = 10^3 \times \begin{pmatrix} 1.0005 & 0 & 0 \\ 0.0010 & 1.0005 & 0.0010 \end{pmatrix}, \ M = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$$

The obtained attenuation with the proposed approach is around $10^{(-11)}$ but we deliberately limited it to $\gamma = 10^{(-3)}$ by replacing the condition $\gamma > 0$ by $\gamma > 10^{(-3)}$. The figure 2 illustrates the fault and the associated residual signal obtained with the proposed approach. It can be seen that the fault is well detected with a magnitude close to that of the fault (i.e. 1). Note that the gain matrix obtained by the classical approach has values around 10^6 while those obtained by the proposed approach are around 10^3.

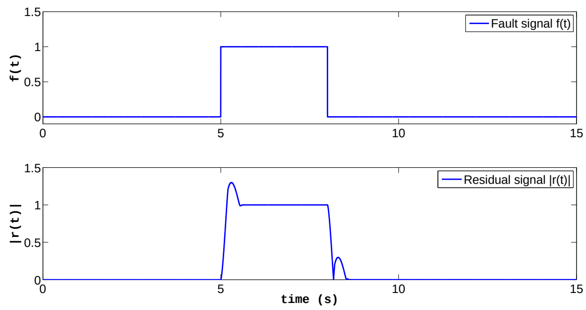

Fig. 2. Fault and residual signal (proposed approach)

In order to test the performances of the proposed approach in noisy measurements, let us consider the same system and add a measurement noise $\omega(t)$ with is a centered random signal and bounded by $[-0.01, 0.01]$. The figure 3 illustrates the residual signals obtained by the two approaches. On can see that, the classical approach provides a residual signals which is masked by the effect of the measurement noise, while the proposed approach provides a residual signal that detects clearly the fault.

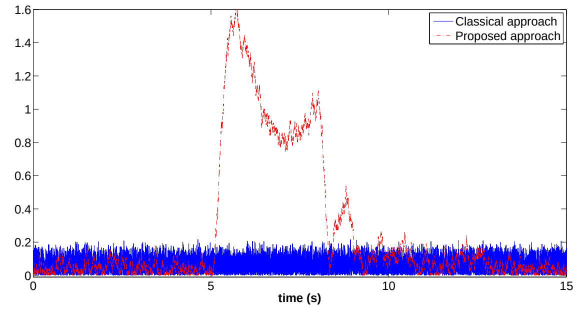

Fig. 3. Comparison of the two approaches with noisy measurements

4.2.0.1. Frequency analysis In is interesting to see the performances of the proposed approach in the frequency domain. For that purpose, let us compute the transfer functions from $f(t)$ to $r(t)$ for the two approaches, it follows

$$H(s) = MC\left(sI - A + LC\right)^{-1}E = \frac{-0.07883}{s^2 + 2.509 + 4.664} \tag{26}$$

and

$$\tilde{H}(s) = \tilde{M}\tilde{C}\left(sI - A + \tilde{L}\tilde{C}\right)^{-1}(E - \tilde{L}\tilde{R}) \approx 1 \tag{27}$$

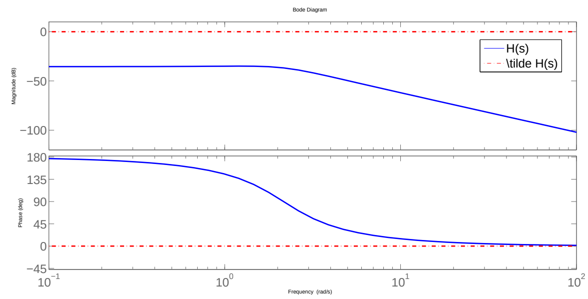

Fig. 4. Frequency analysis of both the classical and proposed RG

From the transfer functions and the Bode diagrams displayed in figure 4, it can be seen that with the classical approach, the transfer from $f(t)$ to $r(t)$ is attenuated while the transfer of the proposed approach is equal to $0dB$ which means that the residual obtained by the proposed approach is close to the fault (this can be seen as a fault estimation, there is a direct transfer from $f(t)$ to $r(t)$).

5. DISCUSSION ON EVENTUAL EXTENSIONS TO SYSTEMS WITH $U(T) \neq 0$

In the context of controlled systems, the control input $u(t)$ is different from zero. In this case the system (7) becomes

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ef(t) \\ y(t) = Cx(t) \end{cases} \tag{28}$$

In order to extend the proposed residual generator, two cases can be considered. If the output has the same relative degree with respect to both $u(t)$ and $f(t)$, the approach is exactly the same. The r^{th} time derivative of $y(t)$ is described by

$$y^{(r)}(t) = CA^r x(t) + CA^{r-1}Bu(t) + CA^{r-1}Ef(t) \tag{29}$$

The new output $\tilde{y}(t)$ is then defined as follows

$$\tilde{y} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^r \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ CA^{r-1}B \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ CA^{r-1}E \end{pmatrix} f \quad (30)$$

Since the control input is known, the residual generator error dynamics coincides exactly with the equation (18), and the same approach is adopted. This is also the case if the relative degree of the output with respect to $f(t)$ is less than the relative degree with respect to the input $u(t)$.

Consider now the case where the relative degree of the output with respect to $f(t)$ is greater than the relative degree with respect to the input $u(t)$. Let us denote r and r_u respectively, the relative degrees with respect to $f(t)$ and $u(t)$ with $r > r_u$. The time derivatives of the output can be written as:

$$\tilde{y} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{r_u} \\ CA^{r_u+1} \\ \vdots \\ CA^r \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ CA^{r_u-1}B \\ CA^{r_u}B \\ \vdots \\ CA^{r-1}B \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ CA^{r_u-1}B \\ \vdots \\ CA^{r-2}B \end{pmatrix} \dot{u} + ...$$

$$+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ CA^{r-2}B \end{pmatrix} u^{(r-r_u)} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ CA^{r-1}B \end{pmatrix} f \quad (31)$$

From this equation, it is necessary to estimate the time derivatives of the control input $u(t)$ which will be used by the residual generator. These time derivatives can be obtained in the same manner as those of $y(t)$. But, it is possible to proceed differently by avoiding the estimation of the time derivatives of the input.

In order to overcome the computation of the time derivatives of the input $u(t)$, let us consider the system (28) and the intermediate system

$$\begin{cases} \dot{s}(t) = As(t) + Bu(t) \\ y_s(t) = Cs(t) \end{cases} \quad (32)$$

By defining the error $z(t) = x(t) - s(t)$, it follows

$$\begin{cases} \dot{z}(t) = Ax(t) + Ef(t) \\ y_z(t) = y(t) - y_s(t) = Cz(t) \end{cases} \quad (33)$$

The system (33) is, then, free from the input $u(t)$ and it is more interesting to construct the residual generator for this system instead of (28). It is then clear that using this intermediate system, the time derivatives of the input are no longer needed.

6. EXTENSION TO MULTI-FAULTS AND MULTI-OUTPUTS SYSTEMS

In this section, an extension of the presented idea to the case of multi-faults and multi-outputs systems is considered. Let us consider the MIMO system

$$\begin{cases} \dot{x}(t) = Ax(t) + Ef(t) \\ y(t) = Cx(t) \end{cases} \quad (34)$$

where $x(t) \in \mathbb{R}^n$, $f(t) \in \mathbb{R}^{n_f}$ and $y(t) \in \mathbb{R}^{n_y}$ are the state, fault and output vectors respectively, and A, E, and C are constant real matrices. The system (34) can be expressed as follows

$$\begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{n_f} E_i f_i(t) \\ y(t) = Cx(t) \end{cases} \quad (35)$$

where $f_i(t),\ i = 1, ..., n_f$ are the components of the vector $f(t)$ with the associated vectors E_i which are the different columns of E. By defining the new output vector

$$\tilde{y}(t) = \tilde{C}x(t) + \tilde{R}f(t) \quad (36)$$

where

$$\tilde{C} = \begin{pmatrix} C_1 \\ C_1A \\ \vdots \\ C_1A^{r_1} \\ C_2 \\ C_2A \\ \vdots \\ C_2A^{r_2} \\ \vdots \\ C_{n_y}A^{r_{n_y}} \end{pmatrix}, \tilde{R} = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ C_1A^{r_1-1}E_1 & \cdots & C_1A^{r_1-1}E_{n_f} \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ C_2A^{r_1-1}E_1 & \cdots & C_1A^{r_1-1}E_{n_f} \\ \vdots & \vdots & \vdots \\ C_{n_y}A^{r_{n_y}-1}E_1 & \cdots & C_{n_y}A^{r_{n_y}-1}E_{n_f} \end{pmatrix}$$

The matrix $C_j,\ j = 1, ..., n_y$ denotes the j^{th} row of the matrix C. The system with the new output vector is then expressed as follows

$$\begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{n_f} E_i f_i(t) \\ \tilde{y}(t) = \tilde{C}x(t) + \tilde{R}f(t) \end{cases} \quad (37)$$

The proposed residual generator is then expressed by

$$\begin{cases} \dot{\hat{x}}(t) = Ax(t) + \tilde{L}\left(\tilde{y}(t) - \hat{\tilde{y}}(t)\right) \\ \hat{\tilde{y}}(t) = \tilde{C}\tilde{x}(t) \\ r(t) = \tilde{M}\left(\tilde{y}(t) - \hat{\tilde{y}}(t)\right) \end{cases} \quad (38)$$

Let us consider the estimation error $e(t) = x(t) - \hat{x}(t)$, its dynamics are described by

$$\begin{cases} \dot{e}(t) = \left(A - \tilde{L}\tilde{C}\right)e(t) + (E - L\tilde{R})f(t) \\ r(t) = \tilde{M}\tilde{C}e(t) + \tilde{M}\tilde{R}f(t) \end{cases} \quad (39)$$

The objective is to make each component of $r(t)$ sensitive to one and one fault of the vector $f(t)$. For that purpose, let us define the virtual residual signal vector, defined by $r_e(t) = r(t) - f(t)$. The global system can be, then, expressed as follows

$$\begin{cases} \dot{e}(t) = \left(A - \tilde{L}\tilde{C}\right)e(t) + (E - L\tilde{R})f(t) \\ r_e(t) = \tilde{M}\tilde{C}e(t) + \left(\tilde{M}\tilde{R} - I_{n_f}\right)f(t) \end{cases} \quad (40)$$

The problem is then transformed into a minimization problem that minimizes the effect of $f(t)$ on $r_e(t)$. In this formulation, each residual signal $r_i(t),\ i = 1, ..., n_f$ is sensitive only to the fault $f_i(t)$. This provides fault isolation (even fault estimation if the transfer is close to zero).

The objective of designing the gains $\tilde{L}$ and $\tilde{M}$ are obtained by solving the optimization problem given in the theorem 2.

Theorem 2. Under the observability of the pair (C, A) and the relative degree vector $\{r_1, ..., r_{n_y}\}$, the residual generator exists if there exist a symmetric and positive definite matrix P, a gain matrix K and a positive scalar γ solution to the following optimization problem

$$\min_{P,K,M} \gamma$$

s.t.

$$\begin{pmatrix} A^T P + PA - \tilde{C}^T \tilde{K}^T - \tilde{K}\tilde{C} & PE - \tilde{K}\tilde{C} & \tilde{C}^T \tilde{M}^T \\ E^T P - \tilde{C}^T \tilde{K}^T & -\gamma I_{n_f} & R^T \tilde{M}^T - I_{n_f} \\ \tilde{M}\tilde{C} & \tilde{M}R - I_{n_f} & -\gamma I_{n_f} \end{pmatrix} < 0 \tag{41}$$

The gain L of the residual generator is obtained from the equation $L = P^{-1}K$. The attenuation level γ describes the sensitivity of $r(t)$ with respect to $f(t)$. The smallest is γ the greatest is the sensitivity.

Proof. The proof is similar the one provided for the theorem 1

7. DISCUSSIONS

Notice that the theorem 2 considers the worst case. However, with a simple analysis on the system (40), it can be concluded that:

(1) If the condition

$$rank\left(\begin{bmatrix} \tilde{C} & \tilde{R} \\ 0 & I_{n_f} \end{bmatrix}\right) = rank\left(\begin{bmatrix} \tilde{C} & \tilde{R} \end{bmatrix}\right) \tag{42}$$

Then there exists a matrix M such that

$$\begin{cases} M\tilde{C} = 0 \\ M\tilde{R} = I_{n_f} \end{cases} \tag{43}$$

Consequently, the system (40) becomes

$$\begin{cases} \dot{x}(t) = \left(A - \tilde{L}\tilde{C}\right) e(t) + (E - L\tilde{R}) f(t) \\ r_e(t) = 0 \end{cases} \tag{44}$$

In this case, it appears that $r_e(t)$ is exactly zero which means that there is a direct transfer from $f(t)$ to $r(t)$ i.e. $r(t) = f(t)$. Therefore, theoretically, the matrix L is not needed, it can be fixed to zero. In addition, the observability and the detectability of the pair $(A, \tilde{C})$ is no longer required. However, in practice, since the time derivatives of the outputs are estimated exactly in finite time $T \neq 0$, the gain L should be computed to ensure stability of the matrix $(A - LC)$ which requires at least the detectability of the pair $(A, \tilde{C})$. But if the matrix A is Hurwitz, the gain L can be fixed to zero.

(2) If the rank condition above is not satisfied but, since $rank(\tilde{R}) = n_f$, then there exist matrices M and L such that $L\tilde{R} = E$ and $M\tilde{R} = I_{n_f}$, and in addition, if the matrix L stabilizes the matrix $A - LC$, the the system (40) becomes

$$\begin{cases} \dot{x}(t) = \left(A - \tilde{L}\tilde{C}\right) e(t) \\ r_e(t) = M\tilde{C}e(t) \end{cases} \tag{45}$$

which is stable, and the virtual residual signal $r_e(t)$ converges asymptotically to zero which means that the real residual vector $r(t)$ converges asymptotically towards $f(t)$.

(3) Finally, if the conditions cited above are not satisfied, the conditions of theorem 2 can be used. This means that there is nor direct fault transfer neither asymptotic fault transfer but only a bounded transfer characterized by the attenuation level γ.

8. CONCLUSION

This paper addresses a new approach for actuator fault detection by using the H_∞ framework combined to the notion of relative degree. It is proven that exploiting the property of the relative degree provides better results compared to the classical H_∞ approach. The idea is to compute the time derivatives of the output signal up to the relative degree, this aims to define a new output vector gathering all the computed time derivatives. Using this reasoning, the limitation $\gamma > 1$ of the classical approach is avoided, and better performances residual generator can be obtained. The approach is motivated by the recent advances in robust numerical derivation of a given signal. Future work will concern, firstly, the extension of the approach for Linear Parameter Varying systems and application for fault tolerant control. Secondly, the approach will be extended for systems affected simultaneously by faults and perturbations.

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