Auxiliary dynamics for observer design of nonlinear TS systems with unmeasurable premise variables

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Abstract: In this paper, the problem of observers for Takagi-Sugeno (TS) models with unmeasurable premise variables is investigated and a new design approach is proposed. The idea, is motivated by the immersion techniques and auxiliary dynamics generation, and consists in the transformation of the TS system with state dependent weighting functions in a new TS system with weighting functions depending only on the measured variables. This result aims to relax the strong conditions used in the design of observers for TS systems with unmeasurable premise variables. Illustrative example is provided to discuss the performances of the proposed approach.

1. INTRODUCTION

From the beginning of the years 2000, the problem of state observer design for Takagi-Sugeno systems having state dependent weighting functions was investigated. Until now, it remains an open problem. The focus is made on the case where the states involved in the weighting functions are not known. The first result was proposed in Bergsten et al. [2001] which extends the Thau-Luenerger observer Thau [1973]. This result has provided LMI conditions depending on the assumption that the perturbation-like term is Lipschitz. The LMIs depend on the Lipschitz constant. This result is extended in Lendek et al. [2009] for TS cascaded systems and in Ichalal et al. [2010] by splitting the perturbation-like term and compute the Lipschitz constants of the weighting functions. The main drawback of these approaches is in the Lipschitz constant LMI-dependent. Indeed, the admissible Lipschitz constant (maximal value of this constant) allowing the existence of a solution to the LMI conditions is often very small which limits the applicability of the Lipschitz approach. Inspired by the differential mean value theorem, used for Lipschitz nonlinear systems Zemouche et al. [2008], a new result is provided by applying this theorem for TS systems with unmeasurable premise variables in Ichalal et al. [2011b], Ichalal et al. [2011a]. The interest of such a result is that it has proposed LMI conditions free from the Lipschitz constant and with asymptotic convergence property. Then, this approach overcomes the limitation related to the Lipschitz assumption and its constant. However, the number of LMIs may become huge which introduce computational complexity. Recently, works have been proposed in order to reduce the conservatism of the Lipschitz approach. The idea is to leave the asymptotic convergence for only bounded error convergence, see for example the quasi Input-to-State Stability (qISS) approach in Ichalal et al. [2012].

Generally, since the linear systems domain is well understood and huge number of algorithms and theories have been developed for control, observation and analysis, the central questions that arise from the beginning of the nonlinear systems story are: Are there coordinate transformations that transform the original system into a linear one in the new coordinates? What are the conditions ensuring the existence of such coordinate transformations? Due to recent developments of some classes of nonlinear systems the first question is relaxed by seeking coordinate transformations that transform the system in nonlinear particular structures, for example, linear systems modulo output injection or state affine systems. In the TS framework, until now, the only used transformation is the sector nonlinearity transformation. Indeed, in the major part of control and observer design, the nonlinear system is kept in the original coordinates and transformed it in a polytopic form (TS form). Especially, in the state observation field, this reasoning introduced several difficulties in stability study of the state estimation error when the premise variables depend of the unmeasured states of the system.

This paper is motivated by the notion of exact linearization without feedback of nonlinear systems introduced in Krener and Isidori [1983] which has aroused great interest as shown by the rich literature in this domain Kazantzis and Kravaris [1998], Glumineau et al. [1996], Keller [1987], Besancon [2007], Souleiman et al. [2003], etc. It introduces a new algorithm for state observer design for TS systems by using the immersion techniques which transforms a TS system with unmeasurable premise variables into a TS system with measurable premise variables in the new coordinates. The proposed immersion algorithm can be viewed as an extension of the state vector with new variables (auxiliary dynamics). Notice that using TS structure reduces the complexity in searching an adequate coordinate transformation compared to the existing approaches which seek of new coordinate transformations in order to have a special nonlinear structure (observable normal forms). In the TS framework, the only objective is to find a LPV system with parameters depending on measured signals, then the LPV system can be transformed into a TS form by using the sector nonlinear transformation in a compact set of the state space.

2. PRELIMINARIES AND PROBLEM STATEMENT

Let us consider the nonlinear system

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)) \tag{1}$$

In the context of Takagi-Sugeno systems, the system (1) can be transformed into a T-S system by using the sector nonlinearity approach Tanaka and Wang [2001] in a compact set of the state space as follows

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(\xi(t)) \left(A_i x(t) + B_i u(t)\right) \\ y(t) = \sum_{i=1}^{r} h_i(\xi(t)) C_i x(t) \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_y}$, $y \in \mathbb{R}^{n_y}$ and A_i, B_i and C_i are known matrices with appropriate dimensions. The weighting functions $h_i(\xi(t))$ depend on the premise variable $\xi(t)$ and satisfy the convex sum property. Firstly, assume that the premise variable $\xi(t)$ depend only on measured signals or signals available at real time. Consequently, the state observer takes the form

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i(\xi(t)) \left(A_i \hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t))\right) \\ \hat{y}(t) = \sum_{i=1}^{r} h_i(\xi(t)) C_i \hat{x}(t) \end{cases} \tag{3}$$

Note that the TS system (2) and the observer (3) share the same premise variables. By defining the state estimation error $e(t) = x(t) - \hat{x}(t)$, its dynamics obeys to the differential equation

$$\dot{e}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\xi(t)) h_j(\xi(t)) \left(A_i - L_i C_j\right) e(t) \tag{4}$$

Notice that the stability study of the TS systems of the form (4) has attracted a lot of attention and several approaches were proposed in order to provide less conservatism LMI conditions Tanaka and Wang [2001], Guerra et al. [2006], Sala and Arinõ [2007] etc.

However, in the case of premise variables depending on unmeasured states $\xi(t) = x(t)$, these approaches cannot be directly exploited. Indeed, in such a case, the observer does not share the same premise variables as the TS system (2) but only the estimated ones as follows

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i(\hat{\xi}(t)) \left(A_i \hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t))\right) \\ \hat{y}(t) = \sum_{i=1}^{r} h_i(\xi(t)) C_i \hat{x}(t) \end{cases} \tag{5}$$

All the techniques dealing with this case considers linear output equation $y(t) = Cx(t)$. With this restriction, the state estimation error dynamics becomes

$$\dot{e}(t) = \sum_{i=1}^{r} h_i(\hat{\xi}(t)) \left(A_i - L_i C\right) e(t) + \delta(t) \tag{6}$$

where $\delta(t) = \sum_{i=1}^{2} \left(h_i(\xi(t)) - h_i(\hat{\xi}(t))\right) \left(A_i x(t) + B_i u(t)\right)$. It is then clear that studying the stability of the system (6) generating the state estimation error is more difficult than that of the system (4). Intensive researches have been devoted to this problem and some results were provided. In Bergsten et al. [2001], a method based on the Lipschitz assumption of the term $\delta(t)$ has been established which ensures asymptotic convergence of the state estimation error toward zero. The same idea were used in Lendek et al. [2009] and Ichalal et al. [2010]. The main drawback of these approaches is that the LMI conditions are feasible only for Lipschitz constants having very small values which limits the domain of applicability of these approaches. In addition, the computation of the Lipschitz constant may become a hard task for complex systems. In order to avoid these problems, an approach based on the mean real value theorem and the sector nonlinearity transformation Ichalal et al. [2011b],Ichalal et al. [2011a]. The main advantage of this approach is the establishment of LMI conditions free form the Lipschitz constant. However, For complex systems it may happen that the number of LMIs becomes huge which limits the existence of a solution, nevertheless, the limitation related to the Lipschitz constant is avoided. Notice that the cited approaches aim to provide conditions ensuring asymptotic or exponential convergence. More recently, new approaches have been proposed by replacing the asymptotic convergence by only the bounded error convergence. It is illustrated that the LMI conditions are relaxed compared the asymptotic approaches, hence, the bounded error property is the price to pay to obtain a solution (see for example, Ichalal et al. [2012] by using the $\mathcal{L}_2$ approach or quasi-Input-to-State Stability (qISS)).

In this paper, a new idea is proposed. it is the answer to the question: Is there a coordinate transformation aiming to transform the TS system with unmeasurable premise variables into a new TS system having weighting functions depending only on measured variables? The present paper addresses an answer based on the immersion techniques and auxiliary dynamics used in some observation and control problems of nonlinear systems. The approach is: from a nonlinear system of the form (1) and before using the sector nonlinear transformation, the auxiliary dynamics are used in order to express the system in a Linear Parameter Varying (LPV) system where the matrices depend only on measured variables. Then, the sector nonlinearity transformation can be used in order to express exactly the new system in TS form. Finally, since the premise variables of the obtained TS model are measured, the classical observer (3) can be used and then exploit the rich algorithms and tools developed for stability analysis of the state estimation error of TS systems with measurable premise variables.

3. MAIN RESULT

The aim of the proposed procedure is to seek for a new coordinate transformation of the nonlinear system in such a way to obtain, after sector nonlinear transformation, a TS model with measurable premise variables, and then the existing results can be exploited, which reduces the design complexity and avoids the additional restrictive assumptions required for the existing methods in the framework of observer for TS systems with unmeasurable premise variables. The classical procedure for observer design of nonlinear systems using TS models is:

(1) **Step 1:** Transform the nonlinear system in a quasi-LPV model
(2) **Step 2:** Using the sector nonlinear transformation, obtain a TS form in a compact set of the state space. If the premise variables are measured or available at real time, go to step 3. If the premise variables are unknown, go to step 4.
(3) **Step 3:** Design the observer (3).
(4) **Step 4:** Design the observer (5).

The proposed procedure is:

(1) **Step 1:** For the nonlinear system (1), obtain new coordinates from the immersion techniques and auxiliary dynamics. The obtained system is made in LPV form where the parameters depend only on measured signals (input and / or output).
(2) **Step 2:** Using the sector nonlinear transformation, obtain a TS form in a compact set of the state space. The premise variables are measured.
(3) **Step 3:** Design the observer (3).

It can be seen, from the above comparison of the classical and the proposed algorithm, that the proposed approach uses only the simple approach with TS models having measured premise variables. Then, the complexity related to the stability analysis of the state estimation error of TS systems with unmeasurable premise variables is avoided.

3.1 Auxiliary dynamics generation

The technique consists in immersing the state space of the original system (1) of dimension n into an other state space of dimension $N \geq n$ by preserving the input-output map. This approach can be seen as the extension of the state vector with new variables coming from the different nonlinearities of the system.

Transformation procedure

(1) **Step 1:** Initialize the first new variables from the state vector $z_i = x_i, i = 1, ..., n$.
(2) **Step 2:** For each new variable, compute its time derivative (exactly the same equations as the original system) and separate all the functions depending only on measured variables y and u and define the remaining nonlinear functions as new variables $z_k, k > n$. By differentiating these variables it follows

$$\dot{z}_k = \frac{\partial z_k}{\partial x}\left(f(x) + g(x)u\right) = \sum_{i=1}^{k} a_{k,i}(u,y)z_i + \sum_{i=k+1}^{l} a_{k+1,i}(u,y)z_i + \varphi_k(u,y) \quad (7)$$

where $z_i, i = k+1, ..., l$ denote other defined new variables. The step 2. is repeated for all the defined variables and the parameter l converges to N the dimension of the new state vector z.
(3) **Step 3:** The algorithm stops when the time derivative of the k^{th} new state is free from nonlinear functions depending on the unknown states.

After computing the system in the new coordinates, it can be expressed as follows

$$\begin{cases} \dot{z} = A(u,y)z + B(u,y)u + \varphi(u,y) \\ y = Cz \end{cases} \quad (8)$$

Finally, by considering the compact set $\mathcal{U} \times \mathcal{Y}$ where $u \in \mathcal{U}$ and $y \in \mathcal{Y}$, the system in TS form is obtained

$$\begin{cases} \dot{z} = \sum_{i=1}^{r} h_i(\xi)\left(A_i z + B_i u\right) + \varphi(u,y) \\ y = Cz \end{cases} \quad (9)$$

where the premise variables depend only of measurable variables y and u. Therefore, the problem of observer design becomes more easier with measurable premise variables compared to the direct design which provides unmeasurable premise variables and leads to complex stability study and conservative results.

Remark 1. By comparison with the use of immersion in nonlinear systems, the target model in the new coordinates takes particular forms such as linear observable form with output injection, observable state affine model with output injection, etc. Notice that the computation of the coordinate transformation may be very difficult for such target models. The proposed algorithm aims to transform the system in less conservative form which is expressed by $\dot{z} = A(u,y)z + B(u,y)u$ with general structure (not necessarily in triangular form) which relaxes the search of the adequate transformation. The obtained form can be easily transformed in TS system with measurable premise variables, then, the existing results of TS systems state observer design can be used and sufficient LMI conditions can be derived easily which are less conservative compared to the LMI conditions obtained directly from the TS system with unmeasurable premise variables which require additional assumptions (Lipschitz condition, boundedness on the nonlinearities and the states,...).

Remark 2. Notice that in the immersion algorithm, the transformation is not unique and for some of them, the properties of the original system may be lost in the new coordinates (especially, in this paper, we speak about observability and detectability). Indeed, there may exit different change of coordinates expressing the system in the form $\dot{z} = A(u,y)z + B(u,y)u$. An analysis should be performed before choosing the most adequate change of coordinates which satisfy observability or at least detectability.

Remark 3. Notice also that the new vector z of the transformed system contains the state x of the original system. This fact is interesting since inverse transformation is not needed in order to recover the state x from z, it is obtained directly from z.

4. SIMULATION EXAMPLE AND COMPARISONS

Let us consider the simple example defined by

$$\dot{x}_1 = x_1 - x_1 x_2^2 + u, \quad \dot{x}_2 = -x_1 - 2x_2, \quad y = x_1 \quad (10)$$

Lipschitz approach Bergsten et al. [2001], Lendek et al. [2009] By applying directly the sector nonlinear transformation by considering the premise variable $\xi(t) =$

$x_1(t)x_2(t)$ and assuming that $d_{\min} \leq \xi(t) \leq d_{\max}$, the TS system is obtained

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \sum_{i=1}^{2} h_i(\xi(t)) \begin{bmatrix}1 & -a_i\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix} u\\ &= \sum_{i=1}^{2} h_i(\xi(t)) A_i x(t) + Bu \qquad (11)\end{aligned}$$

where $h_1(\xi(t)) = \frac{\xi(t)-d_{\min}}{d_{\max}-d_{\min}}, h_2(\xi(t)) = \frac{d_{\max}-\xi(t)}{d_{\max}-d_{\min}}$ and $a_1 = d_{\max}, a_2 = d_{\min}$. The output of the system is linear and described by $y(t) = [1\ \ 0]x(t) = Cx(t)$. Classically, the observer for this system is expressed as follows

$$\dot{\hat{x}}(t) = \sum_{i=1}^{2} h_i(\hat{\xi}(t)) \left(A_i \hat{x}(t) + L_i(y(t) - \hat{y}(t))\right) + Bu \quad (12)$$

Notice that since the premise variable $\xi(t)$ depends on the unknown state $x_2(t)$, it is replaced by its estimation $\hat{\xi}(t)$. By considering the state estimation error $e(t) = x(t) - \hat{x}(t)$, its dynamics is expressed by

$$\dot{e}(t) = \sum_{i=1}^{2} h_i(\xi(t)) \left(A_i - L_i C\right) e(t) + \delta(t) \qquad (13)$$

where

$$\delta(t) = \sum_{i=1}^{2} \left(h_i(\xi(t)) - h_i(\hat{\xi}(t))\right) A_i x(t) = \begin{bmatrix}0 & -(\xi(t) - \hat{\xi}(t))\\ 0 & 0\end{bmatrix} x(t) \qquad (14)$$

Knowing that $\xi(t) = x_1(t)x_2(t)$, it is locally Lipschitz and the Lipschitz condition can be expressed by $\left|\xi(t) - \hat{\xi}(t)\right| < \eta \left\|e(t)\right\|$, and since $x_2(t)$ is bounded $|x_2(t)| < \sigma_2$, the term $\delta(t)$ can be bounded by $\|\delta(t)\| < \gamma \|e(t)\|$, where $\gamma = \eta\sigma_2$ is the Lipschitz constant of $\delta(t)$. From these assumptions, the LMI conditions given in [Bergsten2002], recalled here

$$A_i^T P + P A_i - K_i C - C^T K_i^T < -Q, \quad i = 1, 2 \qquad (15)$$

$$\begin{bmatrix}-Q + \gamma^2 I & P\\ P & -I\end{bmatrix} < 0 \qquad (16)$$

can be solved with symmetric and positive definite matrices P and Q and gain matrices $K_i, i = 1, 2$. If a solution exits, the gains of the observer are deduced from $L_i = P^{-1} K_i, i = 1, 2$. In order to compute the maximal admissible Lipschitz constant, the LMI conditions can be modified and expressed as an optimization problem which maximizes the constant γ and ensuring the asymptotic convergence of the state estimation error. This optimization problem takes the form

$$\min_{P,Q,K_i,\tau} \tau$$

$$A_i^T P + P A_i - K_i C - C^T K_i^T < -Q, \quad i = 1, 2 \qquad (17)$$

$$\begin{bmatrix}-Q & P & I\\ P & -I & 0\\ I & 0 & -\tau I\end{bmatrix} < 0 \qquad (18)$$

After solving this problem, the maximal admissible Lipschitz constant is obtained by $\gamma^* = \frac{1}{\tau}$. It means that if the real Lipschitz constant γ is greater that γ^*, there is no solution to the LMIs and then the observer cannot be designed with such an approach.

For numerical simulations, let us consider the bounded premise variables as follows $-6 \leq \xi(t) \leq -0.5$, this leads to the matrices

$$A_1 = \begin{bmatrix}1 & 0.5\\ -1 & -2\end{bmatrix}, A_2 = \begin{bmatrix}1 & 6\\ -1 & -2\end{bmatrix}$$

and the Lipschitz constant $\gamma = 3.4409$. By using the YALMIP and Sedumi solver, it is found that there is no solution to the LMIs (15)-(16). In addition, the optimization problem (17)-(18) provides the maximal admissible Lipschitz constant $\gamma^* = 2.0616$. If the real Lipschitz constant exceeds the admissible one γ^* then, there is no solution, which is the case for this example. It can be concluded that the Lipschitz approach is conservative and provides solution in a very local state space providing Lipschitz constants less than γ^*.

Mean Value Theorem approach Ichalal et al. [2011b],Ichalal et al. [2011a] For this approach, the same TS model obtained in the previous section is used. The difference is in the treatment of the perturbation-like term $\delta(t)$ (14), this term can be expressed as follows

$$\delta(t) = \begin{bmatrix}-1\\ 1\end{bmatrix} \left(\xi(t) - \hat{\xi}(t)\right) x_2 \qquad (19)$$

Since the nonlinear function $\xi(t) = \xi(x)$ is differentiable and denote $\hat{\xi}(t) = \xi(\hat{x})$, the Mean Value Theorem can be applied for $x \in \mathcal{X}$ ($\mathcal{X}$ is a bounded set) as follows: there exists a constant vector $c \in]x, \hat{x}[$ such that

$$\xi(t) - \hat{\xi}(t) = \xi(x) - \xi(\hat{x}) = \frac{\partial \xi}{\partial x}(c)\,(x - \hat{x}) \qquad (20)$$

where $\frac{\partial \xi}{\partial x} = \begin{bmatrix}\frac{\partial \xi}{\partial x_1} & \frac{\partial \xi}{\partial x_2}\end{bmatrix}$. Then, the term δ becomes

$$\begin{aligned}\delta(t) &= \begin{bmatrix}-x_2 \frac{\partial \xi}{\partial x_1}(c) & -x_2 \frac{\partial \xi}{\partial x_2}(c)\\ 0 & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} \qquad (21)\\ &= \begin{bmatrix}f_1(x_2, c) & f_1(x_2, c)\\ 0 & 0\end{bmatrix} e \qquad (22)\end{aligned}$$

From the assumption that $x \in \mathcal{X}$, x_2 is bounded and we have the following properties $f_i^{\min} \leq f_i(x_2, c) \leq f_i^{\max}, i = 1, 2$. By using the sector nonlinearity transformation on the nonlinearities f_i, the term $\delta(t)$ is expressed by $\delta(t) = \sum_{j=1}^{4} v_i\left(f_1, f_2\right) \mathcal{A}_j e(t)$. Finally, the state estimation error dynamics (13) is expressed by

$$\dot{e} = \sum_{i=1}^{2} \sum_{j=1}^{4} h_i(\hat{\xi}) v_i\left(f_1, f_2\right) \left(A_i + \mathcal{A}_j - L_i C\right) e \qquad (23)$$

The stability analysis of the system (23) is now easier than (13). By using a quadratic Lyapunov function, the LMI conditions can be expressed as follows ($P = P^T > 0$)

$$\left(A_i + \mathcal{A}_j\right)^T P + P\left(A_i + \mathcal{A}_j\right) - C^T K_i^T - K_i C < 0 \quad (24)$$
$$\forall i = 1, 2, \quad \forall j = 1, ..., 4$$

The gains of the observer are then given by $L_i = P^{-1} K_i$. If we consider the same numerical values as previously, namely, $1 \leq x_1 \leq 4$ and $-1.6 \leq x_2 \leq -0.5$, the bounds on the functions f_1 and f_2 can be computed as follows

$$-2.56 \leq f_1(x_2, c) \leq -0.25, \quad 0.5 \leq f_2(x_2, c) \leq 6.4 \quad (25)$$

A solution is then obtained for the LMIs and the observer gains are computed which ensure asymptotic convergence. Notice that even if the compact set $\mathcal{X}$ is enlarged a solution is provided which relaxes the Lipschitz approach. However, the number of LMIs to solve is 8.

quasi-ISS approach Ichalal et al. [2012] With the quasi-ISS approach, the same observer is considered and the state estimation error (13) is then obtained. The difference with the Lipschitz and MVT approaches is that the term $\delta(t)$ is considered as an external bounded perturbation. Indeed, in the compact set $\mathcal{X}$, this term is bounded since x is bounded. From the result in Ichalal et al. [2012] and by considering a quadratic Lyapunov function, the objective is to solve an optimization problem under LMI conditions. This algorithm provides also a solution and the gains L_i are computed which ensure ISS property instead of asymptotic convergence. The number of LMIs to solve in this case is 3.

Proposed approach with auxiliary dynamics Let us go back to the first example (10). Before transforming this system in TS form thanks to the sector nonlinear transformation, the proposed algorithm is first applied as discussed above.

(1) Step 1: consider the new variables $z_1 = x_1$ and $z_2 = x_2$.
(2) Step 2: The time derivatives of z_1 and z_2 are given by

$$\begin{aligned} \dot{z}_1 &= x_1 - x_1 x_2^2 + u = z_1 - y \underbrace{x_2^2}_{z_3} + u \\ \dot{z}_2 &= -x_1 - 2x_2 = -z_1 - 2z_2 \end{aligned} \tag{26}$$

From the above equation, a new variable is defined $z_3 = x_2^2$.
(3) Step 3: By differentiating z_3 with respect to time, one obtains

$$\dot{z}_3 = 2x_2\dot{x}_2 = -2x_1x_2 - 4x_2^2 = -2yz_2 - 4z_3 \tag{27}$$

Since there is no new variable to define, the algorithm ends.

The obtained change of coordinates is $(z_1(t), z_2(t), z_3(t)) = (x_1(t), x_2(t), x_2^2(t))$ which leads to the extended system

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \dot{z}_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -y(t) \\ -1 & -2 & 0 \\ 0 & -2y(t) & -4 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \tag{28}$$

It can be seen that the state matrix of the obtained system depends only on the measured output $y(t)$. Now, by using the sector nonlinearity transformation, the TS system is obtained

$$\dot{z}(t) = \sum_{i=1}^{2} h_i(\xi(t)) A_i z(t) + Bu \tag{29}$$

where the premise variable is nothing than the measured output $\xi(t) = y(t)$. Under the assumption that $y_{\min} \leq y(t) \leq y_{\max}$, the matrices of the new TS system are given by

$$A_1 = \begin{bmatrix} 1 & 0 & -y_{\max} \\ -1 & -2 & 0 \\ 0 & -2y_{\max} & -4 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 0 & -y_{\min} \\ -1 & -2 & 0 \\ 0 & -2y_{\min} & -4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and the weighting functions are $h_1(\xi(t)) = \frac{\xi(t) - y_{\min}}{y_{\max} - y_{\min}}$, $h_2(\xi(t)) = \frac{y_{\max} - \xi(t)}{y_{\max} - y_{\min}}$, where $\xi(t) = y(t)$ which is completely measurable. The output of the system is given by $y = [\,1\ 0\ 0\,]\, z = Cz$. An observer for this new system is expressed by

$$\dot{\hat{z}}(t) = \sum_{i=1}^{2} h_i(\xi(t)) \left(A_i z(t) + L_i(y(t) - \hat{y}(t))\right) + Bu \tag{30}$$

The state estimation error $e(t) = z(t) - \hat{x}(t)$ obeys to the simple differential equation

$$\dot{e}(t) = \sum_{i=1}^{2} h_i(\xi(t)) \left(A_i - L_i C\right) e(t) \tag{31}$$

Its stability can be studied by different approaches provided in the literature. Notice that, for this example, after estimating the state $z(t)$, the original state $x(t)$ is obtained directly form $\hat{z}(t)$ because $\hat{x}_1(t) = \hat{z}_1(t)$ and $\hat{x}_2(t) = \hat{z}_2(t)$ (There is no need to inverse transformation). With comparison to the Lipschitz-based approach, the present approach overcomes the computation on the Lipschitz constant and reduces the conservatism related to the corresponding LMIs.

With the same numerical example as for the Lipschitz approach, the output $y(t)$ is bounded as follows $0.9 \leq y(t) \leq 4$. The proposed approach provides a solution for the LMIs obtained by a quadratic Lyapunov function $V(e(t)) = e^T(t)Pe(t)$ and recalled here

$$A_i^T P + PA_i - K_i C - C^T K_i^T < 0, i = 1, 2 \tag{32}$$

and the gains of the observer are obtained by $L_i = P^{-1}K_i$ and given by

$$L_1 = \begin{bmatrix} 0.0803 \\ -2.647 \\ -25.098 \end{bmatrix}, L_2 = \begin{bmatrix} 1.4382 \\ -0.1145 \\ -4.3006 \end{bmatrix}$$

For comparison, it can be seen that the proposed approach based on auxiliary dynamics performs better than the three classical approaches (Lipschitz, MVT, qISS). In terms of LMIs, only 2 LMIs are needed. In addition, asymptotic convergence is obtained compared to the qISS approach where only bounded error is guaranteed. Finally, in the previous approaches, high gains may be obtained since the objective is to counteract the nonlinear term $\delta(t)$ while in the proposed approach the gains are not high because there is no longer the nonlinear term $\delta(t)$ to counteract.

5. FURTHER STUDY ON STATE EXTENSION

The main result of this paper aims to extend the state dynamics in order to obtain a new system with state dimension greater than the dimension of the original system. The objective is to obtain a TS system having measured premise variables. But it may happen that the procedure does not provide a new system in finite iteration. For details, let us consider the system

$$\dot{x}_1 = x_1 x_2 \quad \dot{x}_2 = -x_1 - x_2^2, \quad y = x_1 \tag{33}$$

It is clear that if the sector nonlinear transformation is used directly, the obtained TS system has premise variable depending on x_2 which is unmeasured. By applying the proposed procedure, it can be seen that the number of iterations is infinite which means that there is no solution. In order to handle such a problem let us consider the first step of the algorithm which leads to $(z_1, z_2) = (x_1, x_2)$, the time derivative of z_1 gives $\dot{z}_1 = yz_2$. For this first equation, it is in the adequate form. For z_2 one obtains $\dot{z}_2 = -z_1 - z_2^2$, it is necessary to define a new variable $z_3 = z_2^2$ as suggested by the algorithm. By differentiating this new variable it follows

$$\dot{z}_3 = -2z_1 z_2 - 2z_2^3 = -2yz_2 - 2z_2^3 \tag{34}$$

One has to define a new variable $z_4 = z_2^3$. However, the number of variable to be defined is infinite. In order to overcome this problem, the second equation of the system is modified equivalently as follows $\dot{z}_2 = -z_1 - \varphi(y)\frac{z_2^2}{\varphi(y)}$, where $\varphi(y)$ is a function which depends only on measured variable y. Then, the new variable to defined is $z_3 = \frac{z_2^2}{\varphi(y)}$ instead of $z_3 = z_2^2$. By choosing $\varphi(y) = \alpha z_1^\beta$, the time derivative of z_3 leads to

$$\dot{z}_3 = \frac{-2\alpha z_2 z_1^{\beta+1} - (2\alpha + \alpha\beta)\, z_1^\beta z_2^3}{\left(\alpha z_1^\beta\right)^2} \tag{35}$$

It is then easy to compute the variables α and β annihilating the term z_2^3 which cause the infinite iteration. For this example, with $\alpha = 1$ and $\beta = -2$, one obtains $\dot{z}_3 = -2z_1^3 z_2 = -2y^3 z_2$. Then, with one iteration the algorithm gives the following system

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & y & 0 \\ -1 & 0 & -\dfrac{1}{y^2} \\ 0 & -2y^3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{36}$$

Notice that this system is defined for $y \neq 0$ and the original system is not observable for $y \neq 0$. The complexity in this approach is to find the adequate function $\varphi(y)$ because it is not unique, the only objective to design this function is to annihilate the eventual variables causing the infinite iteration of the algorithm.

6. CONCLUSION

The problem of observer design for nonlinear systems via TS systems is investigated. It is illustrated that before transforming the original nonlinear system in a TS form, it is more interesting to use the immersion techniques and auxiliary dynamics in order to avoid the state dependent premise variables. Indeed, using directly the sector nonlinear transformation often leads to TS systems with unmeasurable premise variables i.e. premise variables depending of the unmeasured states of the system. An algorithm is proposed by immersing the original nonlinear system in a new system with dimension greater than or equal to the dimension of the original state space. The input-output map is preserved with the coordinate transformation. Then, the obtained new system is transformed, equivalently, in TS system with measurable premise variables.

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