A new Minimum Variance Observer for Stochastic LPV systems with Unknown Inputs ⋆

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** *PERSEUS Project (CNES)*

Abstract: This paper is dedicated to the design of a state estimator for discrete-time Linear Parameter Varying (LPV) systems affected by unknown inputs and random Gaussian noises. Contrary to the existing work, the observer designed in this paper takes measures at several time steps into account in order to improve the performance (in terms of minimizing the variance estimation error). This approach is based on combining the classical Kalman Filter with the design strategies of deterministic observer for LPV systems in deterministic framework. Then, as an extension of this result, the observer is used for estimation of LPV systems without unknown inputs when state noises have a very high variance in comparison to the measurement noises. Simulation results are presented to illustrate the effectiveness of the proposed approach.

Keywords: LPV Systems, Unknown Input Observer, Kalman Filter, Dynamical Systems, Discrete-time Systems

1. INTRODUCTION

State estimation and filtering have attracted great attention from researchers and engineers due to their importance in automatic control, fault diagnosis, monitoring and supervision of industrial processes. In the context of deterministic systems, different observers have been proposed in order to estimate the state of a system from a set of input-output measurements. See for example the well-known Luenberger observer presented in Luenberger (1971) which is considered among the first techniques for deterministic state estimation. This result has been developed for a linear system and extended for various classes of dynamical systems such as singular systems Koenig (2005), Lipschitz nonlinear systems Koenig (2006), Pertew et al. (2005), Boulkroune et al. (2013), Takagi-Sugeno systems Lendek et al. (2010), Ichalal et al. (2009), Linear Parameter Varying (LPV) systems, etc.

In stochastic framework, dynamical systems are affected by stochastic noises and perturbations having some properties such as white noises. Estimating the state of the system requires the filtration of the measurement in order to estimate the state by minimizing the effect of these stochastic noises. For such systems, the first work dealing with the problem of state estimation is proposed in Kalman (1960). The Kalman Filter (KF) has been used in many applications and has proved its efficiency in filtering and state estimation. For nonlinear systems, the Extended Kalman Filter (EKF) has been developed in order to handle the nonlinearities in a dynamical system affected by stochastic noises. It consists in linearizing the system along the trajectory of the state at each sample time. For linear time-invariant (LTI) and linear time-varying (LTV) systems affected by noises and unknown inputs, a KF has been proposed in Darouach et al. (1995) and Su et al. (2015) in order to estimate optimally the state. A sub-optimal solution has been also provided in Darouach et al. (2003).

In the context of Linear Parameter Varying (LPV) systems affected by unknown inputs, different observers have been proposed in the deterministic framework, one can cite Ichalal and Mammar (2015), Briat et al. (2011), Marx et al. (2007), Fiacchini and Millerioux (2013), etc. LPV represent a large class of dynamical systems in a simple form that can exploit the rich tools and theories developed for linear systems. Notice that LPV systems are, in some way, different from the well-known LTV systems since the parameters may depend on internal or external variables having certain properties (inputs, outputs, external parameters such as, for example, time varying longitudinal velocity in vehicle lateral dynamics model). Furthermore, the varying parameters of an LPV system may have interesting properties such as boundedness, differentiability (or not),...etc.

In this paper, we explore the design of a Minimum Variance Observer (MVO) for discrete-time LPV systems affected by unknown inputs and Gaussian white noises. In that field of research Darouach and Zasadzinski (1997) have first proposed an observer which is efficient with LTV

⋆ This work is supported by the CNES (Centre National d'Etudes spatiales), France.

systems affected by an unknown input in the state equation. Darouach et al. (2003) generalized that work by the development of an observer dealing with the presence of an unknown input in both state and measurement equations. Then, Gillijns and Moor (2007a) has proposed an observer with dual state and unknown input estimation based on a three steps filter, whereas Gillijns and Moor (2007b) generalized the result in the case of the presence of the unknown input in the measurement equation. However, in that last work, it has been assumed that the matrix associated with the unknown input in the measuement equation was full column rank. That last condition has finally been relaxed in Hsieh (2009) and in Yong et al. (2016). In all those previous papers, the idea was to estimate the state at the current time step, using the measurement available at that time step and the estimation at the previous time step. The originality of our paper is to use both measurement at previous and current time step. This approach gives us more information, and also more degree of freedom in order to build a more powerful observer.

The paper is divided as follow. After introducing the problem in section 2, section 3 presents the main result of the work. Section 4 gives an application of the result for linear stochastic systems with high variance in the state equation. Examples are given for both results in order to illustrate the talk.

2. PRELIMINARY, NOTATIONS AND PROBLEM STATEMENT

Let us consider the Linear Parameter Varying (LPV) system expressed by

$$\begin{cases} \mathbf{x}_{k+1} &= A_{\rho_k}\mathbf{x}_k + D_{\rho_k}\mathbf{d}_k + F_{\rho_k}\mathbf{w}_k \\ \mathbf{y_k} &= C_{\rho_k}\mathbf{x}_k + E_{\rho_k}\mathbf{d}_k + \mathbf{v}_k \end{cases} \quad (1)$$

where $\mathbf{x}_k \in \mathbb{R}^{n_x}$, $\mathbf{y_k} \in \mathbb{R}^{n_y}$, and $\mathbf{d}_k \in \mathbb{R}^{n_d}$ are the state vector, the output, and the unknown input of the system. A_{ρ_k}, D_{ρ_k}, F_{ρ_k}, C_{ρ_k} and E_{ρ_k} are parameter-varying matrices with appropriate dimensions. These matrices depend on the bounded parameter varying vector $\rho_k \in \Theta \subset \mathbb{R}^{n_\rho}$. $v_k \in \mathbb{R}^{n_v}$ and $w_k \in \mathbb{R}^{n_y}$ are two independent zero-mean Gaussian white noises with constant covariance matrices W and V respectively, i.e. $W = E[\mathbf{w}_k\mathbf{w}_k^T]$ and $V = E[\mathbf{v}_k\mathbf{v}_k^T]$ for all k.
Throughout all this paper, it is assumed that $n_y \geq n_d$, and that V is non singular ($det(V) \neq 0$).

In this paper, M^T denotes the transpose matrix of M, M^+ its pseudo-inverse, and if M is a square matrix, $tr(M)$ denotes its trace (sum of the diagonal components). I_n denotes the identity matrix of dimension n.

The problem that will be discussed in this paper is the estimation of the state vector $\mathbf{x}_k$ of system (1) from a set of noisy output measurements, independently of the unknown input $\mathbf{d}_k$. The observer that is looking for has to be unbiased, with minimum variance estimation error.

3. UNBIASED MINIMUM VARIANCE OBSERVER

According to our knowledge observers developped to deal with state estimation of system such that (1) use either the measure $\mathbf{y}_k$ or the measure $\mathbf{y}_{k+1}$ in order to estimate $\mathbf{x}_{k+1}$ from the estimation of $\mathbf{x}_k$, but never both measures. In this work both measures are used, and thus a degree of freedom is added for the gain matrices calculation. This approach leads to better results than those obtained in previous works. Furthermore the present work takes place in the context of LPV systems, which is a more general class of system than the LTV, which was mainly studied in previous papers.

3.1 Main result

The main result of this paper is developed in the following theorem.

Theorem 1. Under the following condition:

- $rank\left(\begin{bmatrix} E_{\rho_k} & 0_{n_y\times n_d} \\ C_{\rho_{k+1}}D_{\rho_k} & E_{\rho_{k+1}} \end{bmatrix}\right) = rank\left(\begin{bmatrix} D_{\rho_k} \\ E_{\rho_k} \end{bmatrix}\right) + rank\left(E_{\rho_{k+1}}\right)$, $\forall \rho_k \in \Theta$,

the following equations provide an *unbiased* state estimator with *minimum variance* for the state estimation of the stochastic LPV system of equation (1):

$$\begin{cases} \hat{\mathbf{x}}_{k+1} &= N_{\rho_k}\hat{\mathbf{x}}_k + Q_{\rho_k}\mathbf{y}_k + R_{\rho_k}\mathbf{y}_{k+1} \\ P_{k+1} &= (A_{\rho_k} - L_{\rho_k}\mathcal{C}_{\rho_k})P_k(A_{\rho_k} - L_{\rho_k}\mathcal{C}_{\rho_k})^T \\ &\quad +(F_{\rho_k} - L_{\rho_k}\mathcal{S}_{\rho_k})W(F_{\rho_k} - L_{\rho_k}\mathcal{S}_{\rho_k})^T \\ &\quad +L_{\rho_k}\mathcal{V}L_{\rho_k}^T \\ N_{\rho_k} &= A_{\rho_k} - Q_{\rho_k}C_{\rho_k} - R_{\rho_k}C_{\rho_{k+1}}A_{\rho_k} \\ \left[Q_{\rho_k} \;\; R_{\rho_k}\right] &= L_{\rho_k} \\ L_{\rho_k} &= \mathcal{F}_{\rho_k} + Z_k\mathcal{G}_{\rho_k} \\ Z_k &= ((A_{\rho_k} - \mathcal{F}_{\rho_k}\mathcal{C}_{\rho_k})P_k\mathcal{C}_{\rho_k}^T\mathcal{G}_{\rho_k}^T \\ &\quad +(F_{\rho_k} - \mathcal{F}_{\rho_k}\mathcal{S}_{\rho_k})W\mathcal{S}_{\rho_k}^T\mathcal{G}_{\rho_k}^T \\ &\quad -\mathcal{F}_{\rho_k}\mathcal{V}\mathcal{G}_{\rho_k}^T) \\ &\quad \times(\mathcal{G}_{\rho_k}\mathcal{C}_{\rho_k}P_k\mathcal{C}_{\rho_k}^T\mathcal{G}_{\rho_k}^T \\ &\quad +\mathcal{G}_{\rho_k}\mathcal{S}_{\rho_k}V\mathcal{S}_{\rho_k}^T\mathcal{G}_{\rho_k}^T \\ &\quad +\mathcal{G}_{\rho_k}\mathcal{V}\mathcal{G}_{\rho_k}^T)^{-1} \end{cases}, \quad (2)$$

where

$$\mathcal{F}_{\rho_k} = \mathcal{D}_{\rho_k}\mathcal{A}_{\rho_k}^+, \quad \mathcal{G}_{\rho_k} = M_{\rho_k}(I - \mathcal{A}_{\rho_k}\mathcal{A}_{\rho_k}^+)$$

$$\mathcal{D}_{\rho_k} = [D_{\rho_k} \;\; 0_{n_x,n_d}], \quad \mathcal{A}_{\rho_k} = \begin{bmatrix} E_{\rho_k} & 0_{n_y,n_d} \\ C_{\rho_{k+1}}D_{\rho_k} & E_{\rho_{k+1}} \end{bmatrix}$$

$$\mathcal{C}_{\rho_k} = \begin{bmatrix} C_{\rho_k} \\ C_{\rho_{k+1}}A_{\rho_k} \end{bmatrix}, \quad \mathcal{S}_{\rho_k} = \begin{bmatrix} 0_{n_y,n_v} \\ C_{\rho_{k+1}}F_{\rho_k} \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix},$$

and $M_{\rho_k} = \begin{bmatrix} 0 & I_{2n_y-\gamma_k} \end{bmatrix} U_{\rho_k}^T$, where $\mathcal{A}_{\rho_k} = U_{\rho_k}\begin{bmatrix} \Gamma_{\rho_k} & 0 \\ 0 & 0 \end{bmatrix} T_{\rho_k}^T$ is the Singular Value Decomposition of $\mathcal{A}_{\rho_k}$, Γ_{ρ_k} being a non singular matrix, and $\gamma_k = rank(\mathcal{A}_{\rho_k}) = rank(\Gamma_{\rho_k})$.

Proof. An unbiased observer with minimum variance which uses $\mathbf{y}_k$ and $\mathbf{y}_{k+1}$ can be written under the form:

$$\hat{\mathbf{x}}_{k+1} = N_{\rho_k}\hat{\mathbf{x}}_k + Q_{\rho_k}\mathbf{y}_k + R_{\rho_k}\mathbf{y_{k+1}} \quad (3)$$

The aim of the following proof is to find N_{ρ_k}, Q_{ρ_k} and R_{ρ_k} such that both following conditions are satisfied.

- The observer has to be unbiased:
$$E[\mathbf{e}_k] = 0, \quad (4)$$
where $\mathbf{e}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k$ is the estimation error.
- The observer has to be the minimum variance observer among all unbiased ones, i.e. it has to minimizes $tr(P_k)$, where $P_k = E[\mathbf{e}_k\mathbf{e}_k^T]$ is the error estimation covariance matrix.

Note that the first part of the proof is quite similar to the one proposed in Darouach et al. (2003). Let start by establishing the dynamic of the estimation error

$$\begin{aligned}
&\mathbf{e}_{k+1} = \mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1} \\
&= (A_{\rho_k} - Q_{\rho_k}C_{\rho_k} - R_{\rho_k}C_{\rho_{k+1}}A_{\rho_k})\mathbf{e}_k \\
&+(A_{\rho_k} - Q_{\rho_k}C_{\rho_k} - R_{\rho_k}C_{\rho_{k+1}}A_{\rho_k} - N_{\rho_k})\hat{\mathbf{x}}_k \\
&+(D_{\rho_k} - Q_{\rho_k}E_{\rho_k} - R_{\rho_k}C_{\rho_{k+1}}D_{\rho_k})\mathbf{d}_k - R_{\rho_k}E_{\rho_{k+1}}\mathbf{d}_{k+1} \\
&+(F_{\rho_k} - R_{\rho_k}C_{\rho_{k+1}}F_{\rho_k})\mathbf{w}_k - Q_{\rho_k}\mathbf{v}_k - R_{\rho_k}\mathbf{w}_{k+1}.
\end{aligned} \tag{5}$$

Recalling that $\mathbf{w}_k$, $\mathbf{v}_k$, $\mathbf{w}_{k+1}$ are zero mean noises, the estimation error expectancy can be calculated as follows

$$\begin{aligned}
&E[\mathbf{e}_k] = (A_{\rho_k} - Q_{\rho_k}C_{\rho_k} - R_{\rho_k}C_{\rho_{k+1}}A_{\rho_k})E[\mathbf{e}_k] \\
&+(A_{\rho_k} - Q_{\rho_k}C_{\rho_k} - R_{\rho_k}C_{\rho_{k+1}}A_{\rho_k} - N_{\rho_k})\hat{\mathbf{x}}_k \\
&+(D_{\rho_k} - Q_{\rho_k}E_{\rho_k} - R_{\rho_k}C_{\rho_{k+1}}D_{\rho_k})\mathbf{d}_k - R_{\rho_k}E_{\rho_{k+1}}\mathbf{d}_{k+1}.
\end{aligned} \tag{6}$$

Thus, condition (4) is satisfied if and only if all the following equations are satisfied

$$\begin{cases} 0 &= A_{\rho_k} - Q_{\rho_k}C_{\rho_k} - R_{\rho_k}C_{\rho_{k+1}}A_{\rho_k} - N_{\rho_k} \\ 0 &= D_{\rho_k} - Q_{\rho_k}E_{\rho_k} - R_{\rho_k}C_{\rho_{k+1}}D_{\rho_k} \\ 0 &= R_{\rho_k}E_{\rho_{k+1}} \end{cases} \tag{7}$$

and it can be deduced that $N_{\rho_k} = A_{\rho_k} - Q_{\rho_k}C_{\rho_k} - R_{\rho_k}C_{\rho_{k+1}}A_{\rho_k}$ and that $L_{\rho_k}\mathcal{A}_{\rho_k} = \mathcal{D}_{\rho_k}$, where $L_{\rho_k} = [Q_{\rho_k}\ R_{\rho_k}]$, $\mathcal{A}_{\rho_k} = \begin{bmatrix} E_{\rho_k} & 0_{n_y\times n_d} \\ C_{\rho_{k+1}}D_{\rho_k} & E_{\rho_{k+1}} \end{bmatrix}$ and $\mathcal{D}_{\rho_k} = [D_{\rho_k}\ 0_{n_x\times n_d}]$. The equation $L_{\rho_k}\mathcal{A}_{\rho_k} = \mathcal{D}_{\rho_k}$ has a solution for L_{ρ_k} if and only if the following rank condition is satisfied

$$rank(\mathcal{A}_{\rho_k}) = rank\left(\begin{bmatrix}\mathcal{D}_{\rho_k} \\ \mathcal{A}_{\rho_k}\end{bmatrix}\right), \forall \rho_k \in \Theta \tag{8}$$

which is equivalent to

$$rank(\mathcal{A}_{\rho_k}) = rank\left(\begin{bmatrix}D_{\rho_k} \\ E_{\rho_k}\end{bmatrix} + rank(E_{\rho_{k+1}}\right), \forall \rho_k \in \Theta \tag{9}$$

which corresponds to the rank condition expressed in theorem 1. Indeed:

$$\begin{aligned}
rank\left(\begin{bmatrix}\mathcal{D}_{\rho_k} \\ \mathcal{A}_{\rho_k}\end{bmatrix}\right) &= rank\left(\begin{bmatrix} D_{\rho_k} & 0 \\ E_{\rho_k} & 0 \\ C_{\rho_{k+1}}D_{\rho_k} & E_{\rho_{k+1}} \end{bmatrix}\right) \\
&\times rank\left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ C_{\rho_{k+1}} & 0 & I \end{bmatrix}\begin{bmatrix} D_{\rho_k} & 0 \\ E_{\rho_k} & 0 \\ 0 & E_{\rho_{k+1}} \end{bmatrix}\right)
\end{aligned} \tag{10}$$

(see Darouach et al. (2003) for more details).

The rest of the proof differs from Darouach et al. (2003). The solution L_{ρ_k} can then be written as

$$L_{\rho_k} = \mathcal{F}_{\rho_k} + Z_k\mathcal{G}_{\rho_k}, \tag{11}$$

where Z_k is a gain matrix that will be calculated later, $\mathcal{F}_{\rho_k} = \mathcal{D}_{\rho_k}\mathcal{A}^+_{\rho_k}$, $\mathcal{G}_{\rho_k} = M_{\rho_k}(I_{2n_y\times 2n_y} - \mathcal{A}_{\rho_k}\mathcal{A}^+_{\rho_k})$, and M_{ρ_k} is any coefficient matrix whose choice will be discussed later. By setting $\mathcal{C}_{\rho_k} = \begin{bmatrix} C_{\rho_k} \\ C_{\rho_{k+1}}A_{\rho_k} \end{bmatrix}$ and $\mathcal{S}_{\rho_k} = \begin{bmatrix} 0_{n_y,n_v} \\ C_{\rho_{k+1}}F_{\rho_k} \end{bmatrix}$, the estimation error dynamic becomes

$$\begin{aligned}
&\mathbf{e}_{k+1} = (A_{\rho_k} - L_{\rho_k}\mathcal{C}_{\rho_k})\mathbf{e}_k + (F_{\rho_k} - L_{\rho_k}\mathcal{S}_{\rho_k})\mathbf{w}_k \\
&-L_{\rho_k}\begin{bmatrix} I_{n_y,n_y} \\ 0_{n_y,n_y}\end{bmatrix}\mathbf{v}_k - L_{\rho_k}\begin{bmatrix} 0_{n_y,n_y} \\ I_{n_y,n_y}\end{bmatrix}\mathbf{w}_{k+1},
\end{aligned} \tag{12}$$

with the following expectancy:

$$E[\mathbf{e}_{k+1}] = (A_k - L_{\rho_k}\mathcal{C}_{\rho_k})E[\mathbf{e}_k], \tag{13}$$

which makes the estimator unbiased.

In order to achieve the second condition (the minimum variance estimation), the next step of the estimator construction consists in establishing the covariance matrix evolution of the estimation error $\mathbf{e}_k$. In order to do that, let us consider the equation (12). In that equation, $\mathbf{w}_k$, $\mathbf{v}_k$ and $\mathbf{w}_{k+1}$ are three independent noises, but $\mathbf{e}_k$ and $\mathbf{w}_k$ are not independent (it is obviously seen by rewriting equation (12) at step time $k-1$). However for the proof this dependency will be neglected. A discussion about this approximation will be presented later in the paper. Thus, by setting $\mathcal{V} = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix}$, the calculation of the dynamic equation of the error covariance matrix $P_k = E[\mathbf{e}_k\mathbf{e}_k^T]$ gives:

$$\begin{aligned}
&P_{k+1} \\
&= (A_k - \mathcal{F}_{\rho_k}\mathcal{C}_{\rho_k} - Z_k\mathcal{G}_{\rho_k}\mathcal{C}_{\rho_k})P_k(A_k - \mathcal{F}_{\rho_k}\mathcal{C}_{\rho_k} - Z_k\mathcal{G}_{\rho_k}\mathcal{C}_{\rho_k})^T \\
&+(F_{\rho_k} - \mathcal{F}_{\rho_k}\mathcal{S}_{\rho_k} - Z_k\mathcal{G}_{\rho_k}\mathcal{S}_{\rho_k})W(F_{\rho_k} - \mathcal{F}_{\rho_k}\mathcal{S}_{\rho_k} - Z_k\mathcal{G}_{\rho_k}\mathcal{S}_{\rho_k})^T \\
&+(\mathcal{F}_{\rho_k} + Z_k\mathcal{G}_{\rho_k})\mathcal{V}(\mathcal{F}_{\rho_k} + Z_k\mathcal{G}_{\rho_k})^T.
\end{aligned} \tag{14}$$

Z_k is then calculated in order to minimize $tr(P_{k+1})$. Both first and second derivatives of $tr(P_{k+1})$ with respect to Z_k are

$$\begin{aligned}
&\frac{\partial tr(P_{k+1})}{\partial Z_k} \\
&= 2Z_k(\mathcal{G}_{\rho_k}\mathcal{C}_{\rho_k}P_k\mathcal{C}^T_{\rho_k}\mathcal{G}^T_{\rho_k} \\
&+\mathcal{G}_{\rho_k}\mathcal{S}_{\rho_k}W\mathcal{S}^T_{\rho_k}\mathcal{G}^T_{\rho_k} + \mathcal{G}_{\rho_k}\mathcal{V}\mathcal{G}^T_{\rho_k}) \\
&-2((A_k - \mathcal{F}_{\rho_k}\mathcal{C}_{\rho_k})P_k\mathcal{C}^T_{\rho_k}\mathcal{G}^T_{\rho_k} \\
&+(F_{\rho_k} - \mathcal{F}_{\rho_k}\mathcal{S}_{\rho_k})W\mathcal{S}^T_{\rho_k}\mathcal{G}^T_{\rho_k} \\
&-\mathcal{F}_{\rho_k}\mathcal{V}\mathcal{G}^T_{\rho_k}),
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&\frac{\partial^2 tr(P_{k+1})}{\partial Z_k^2} \\
&= 2(\mathcal{G}_{\rho_k}\mathcal{C}_{\rho_k}P_k\mathcal{C}^T_{\rho_k}\mathcal{G}^T_{\rho_k} + \mathcal{G}_{\rho_k}\mathcal{S}_{\rho_k}W\mathcal{S}^T_{\rho_k}\mathcal{G}^T_{\rho_k} \\
&+\mathcal{G}_{\rho_k}\mathcal{V}\mathcal{G}^T_{\rho_k}).
\end{aligned} \tag{16}$$

Thus, $\frac{\partial^2 tr(P_{k+1})}{\partial Z_k^2}$, which is the hessian matrix of the application $Z_k \mapsto tr(P_{k+1})$, is clearly a non negative symmetric matrix. Besides, by assumption of the theorem, W is non singular, and so is $\mathcal{V}$. Thus, if M_{ρ_k} is chosen such that $\mathcal{G}_{\rho_k}$ is full row rank, then the hessian matrix would be symmetric positive definite, and there would exist a unique solution Z_k that minimizes $tr(P_{k+1})$. In order to construct such a matrix $\mathcal{G}_{\rho_k}$, we set $M_{\rho_k} = \left[0_{2n_y-\gamma_k,\gamma_k}\ I_{2n_y-\gamma_k}\right]U^T_{\rho_k}$ (where $\mathcal{A}_{\rho_k} = U_{\rho_k}\begin{bmatrix}\Gamma_{\rho_k} & 0 \\ 0 & 0\end{bmatrix}T^T_{\rho_k}$ is the singular value decomposition of $\mathcal{A}_{\rho_k}$, with Γ_{ρ_k} non singular, and $\gamma_k = rank(\mathcal{A}_{\rho_k}) = rank(\Gamma_{\rho_k}) \le 2n_y$), and then $\mathcal{G}_{\rho_k} = M_{\rho_k}$ is full row rank.

Finally, by equaling $\frac{\partial tr(P_{k+1})}{\partial Z_k}$ to 0, it comes:

$$\begin{aligned}
&Z_k = ((A_k - \mathcal{F}_{\rho_k}\mathcal{C}_{\rho_k})P_k\mathcal{C}^T_{\rho_k}\mathcal{G}^T_{\rho_k} \\
&+(F_{\rho_k} - \mathcal{F}_{\rho_k}\mathcal{S}_{\rho_k})W\mathcal{S}^T_{\rho_k}\mathcal{G}^T_{\rho_k} \\
&-\mathcal{F}_{\rho_k}\mathcal{V}\mathcal{G}^T_{\rho_k}) \\
&\times(\mathcal{G}_{\rho_k}\mathcal{C}_{\rho_k}P_k\mathcal{C}^T_{\rho_k}\mathcal{G}^T_{\rho_k} + \mathcal{G}_{\rho_k}\mathcal{S}_{\rho_k}W\mathcal{S}^T_{\rho_k}\mathcal{G}^T_{\rho_k} \\
&+\mathcal{G}_{\rho_k}\mathcal{V}\mathcal{G}^T_{\rho_k})^{-1},
\end{aligned} \tag{17}$$

which ends the proof.

An algorithm is presented in table 1 in order to make easier the use of the observer presented in theorem 1.

Remark 2. If $E_{\rho_k} = 0$, then the system represented by equation (1) becomes

$$\begin{cases} \mathbf{x}_{k+1} &= A_{\rho_k}\mathbf{x}_k + D_{\rho_k}\mathbf{d}_k + F_{\rho_k}\mathbf{w}_k \\ \mathbf{y_k} &= C_{\rho_k}\mathbf{x}_k + \mathbf{v}_k \end{cases}. \quad (18)$$

In this case, the rank condition of theorem 1 is written $rank(C_{\rho_{k+1}}D_{\rho_k}) = rank(D_{\rho_k})$ for all $\rho_k \in \Theta$, and the only change in the observer is the expression of matrices $\mathcal{A}_{\rho_k}$ and $\mathcal{D}_{\rho_k}$ that becomes $\mathcal{A}_{\rho_k} = C_{\rho_{k+1}}D_{\rho_k}$ and $\mathcal{D}_{\rho_k} = D_{\rho_k}$.

3.2 Numerical comparisons

In this subsection, a comparison between the estimator developed in Darouach et al. (2003) and the one proposed in theorem 1 will be done by numerical simulations. We choose to use the work done in Darouach et al. (2003) as a reference, the observer developed in theorem 1 is (in a certain sense) an extension (or upgrade) of it (see the proof of the theorem). Thus, numerical comparisons will show the level of this upgrade.

In order to do these simulations, time invariant matrices will be chosen (instead of varying parameters matrices as in model (1)). Simulations are made for several values of n_x, n_y and n_d, and in each case, 100 simulations are used to compare both observers. In each simulation, the matrices A, D, F, C, E, W and V are generated randomly, A satisfying the constraint of being a stable matrix, and C, D and E satisfying the rank condition needed for both observers. The unknown input is a constant whose value is generated randomly for each simulation. The convergence is checked after each simulation by numerical verification. In each case, simulations are done until having 100 cases of common convergence (when both observers converge). The Root Mean Square Error (RMSE) is used as a criteria of comparison of the performance.

The results are shown in table 1. It can be seen, that that the proposed observer achieves better results than the one presented in Darouach et al. (2003). In particular, it shows that the approximation made in the proof concerning the dependency between $\mathbf{e}_k$ and $\mathbf{w}_k$ is not absurd. Besides, the number of convergence cases is better in average for the observer developed in this theorem 1 than the one of Darouach et al. (2003).

3.3 Illustrative example

In order to illustrate the presented result in an LPV example, the proposed observer is compared to the observers detailled in Darouach et al. (2003) and Yong et al. (2016) on the following system:

$$\begin{cases} \mathbf{x}_{k+1} &= \begin{bmatrix} 1 & (1+\rho_k)dt \\ -\rho_k dt & 1-2dt \end{bmatrix}\mathbf{x}_k + \begin{bmatrix} \rho_k dt \\ (1+\rho_k)dt \end{bmatrix}\mathbf{d}_k + \mathbf{w}_k \\ \mathbf{y}_k &= \begin{bmatrix} 0.5 & 0 \\ 0.1 & 0.3 \end{bmatrix}\mathbf{x}_k + \begin{bmatrix} \rho_k dt \\ -1 \end{bmatrix}\mathbf{d}_k + \mathbf{v}_k \end{cases}, \quad (19)$$

where $\mathbf{x}_k = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$ is the state vector. v_k and w_k are one dimension zero-mean Gaussian white noises with standard deviation $\sigma_w = [0.8\ 0.8]^T$ and $\sigma_v = [0.5\ 0.5]^T$

Table 1. Numerical Comparison between the observer developed in Darouach et al. (2003) (Observer 1 in the table) and the one established in theorem 1 (Observer 2 in the table). In bracket the number of convergence cases out of the total number of simulations launched.

	RMSE on ...	x_1	x_2	x_3
$n_x = 3$,	Observer 1 (100/2119)	2.07	1.90	2.24
$n_y = 3$, $n_d = 3$	Observer 2 (2119/2119)	1.30	1.25	1.39
$n_x = 3$,	Observer 1 (100/2045)	2.05	1.75	1.79
$n_y = 3$, $n_d = 2$	Observer 2 (2045/2045)	1.28	1.32	1.30
$n_x = 3$,	Observer 1 (100/100)	1.49	1.43	1.43
$n_y = 3$, $n_d = 1$	Observer 2 (100/100)	1.05	1.09	1.11
$n_x = 3$,	Observer 1 (100/2276)	1.59	1.62	1.54
$n_y = 2$, $n_d = 2$	Observer 2 (2276/2276)	1.21	1.17	1.22
$n_x = 3$,	Observer 1 (100/2250)	1.88	1.72	1.77
$n_y = 2$, $n_d = 1$	Observer 2 (2250/2250)	1.20	1.19	1.26
$n_x = 3$,	Observer 1 (100/2098)	1.54	1.64	1.45
$n_y = 1$, $n_d = 1$	Observer 2 (1427/2098)	1.48	1.55	1.37
$n_x = 2$,	Observer 1 (100/313)	1.81	1.92	–
$n_y = 2$, $n_d = 2$	Observer 2 (313/313)	1.26	1.32	–
$n_x = 2$,	Observer 1 (100/384)	1.53	2.00	–
$n_y = 2$, $n_d = 1$	Observer 2 (384/384)	1.22	1.34	–
$n_x = 2$,	Observer 1 (100/311)	3.59	2.91	–
$n_y = 1$, $n_d = 1$	Observer 2 (255/311)	3.49	2.79	–
$n_x = 1$,	Observer 1 (100/100)	1.75	–	–
$n_y = 1$, $n_d = 1$	Observer 2 (100/100)	1.75	–	–

respectively (covariance matrices $W = \begin{bmatrix} 0.64 & 0 \\ 0 & 0.64 \end{bmatrix}$, and $V = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}$ respectively), and are independent to each other. We take $\rho_k = 3cos(k.dt)^2 + 1$. The simulation is launched during 10 seconds with a step time $dt = 0.1s$. The unknown input shown in figure 1 is applied. The initial covariance matrix is taken to be equal to $P_0 = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$, and the initial estimation is taken to $\mathbf{x}_0 = [0\ 0]^T$.

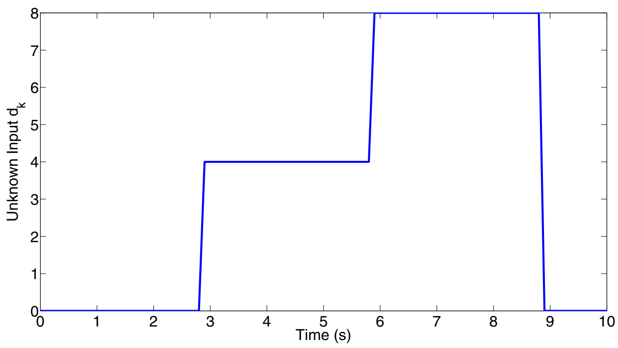

Fig. 1. Unknown Input

The estimations made by the observers provided in Darouach et al. (2003) and in Yong et al. (2016) and by the observer presented in theorem 1 are shown on figure 2 and the associated errors on figure 3.

Figure 4 shows the evolution of the trace of the covariance error matrix P_k. For the observer of theorem 1, the trace is lower to the one obtained by the other observers tested. The sinusoidal evolution after convergence is due to the time variant aspect of matrices used in the example.

The new observer clearly overpasses the others on that

Algorithm 1 Observer algorithm (under the conditions of theorem 1)

Initialization of $\hat{\mathbf{x}}_0$ and P_0. Matrices A_{ρ_k}, D_{ρ_k}, F_{ρ_k}, C_{ρ_k} and E_{ρ_k} for all k, as well as matrices W and V are assumed to be known.

for $k = 0..N$ **do**

$\mathcal{D}_{\rho_k} = [D_{\rho_k} \;\; 0_{n_x,n_d}]$, $\quad \mathcal{A}_{\rho_k} = \begin{bmatrix} E_{\rho_k} & 0_{n_y,n_d} \\ C_{\rho_{k+1}} D_{\rho_k} & E_{\rho_{k+1}} \end{bmatrix}$

$[U_{\rho_k}, \Gamma_{\rho_k}, T_{\rho_k}] = \text{svd}(\mathcal{A}_{\rho_k})$ $^{(1)}$ (such that $\mathcal{A}_{\rho_k} = U_{\rho_k} \begin{bmatrix} \Gamma_{\rho_k} & 0 \\ 0 & 0 \end{bmatrix} T^T_{\rho_k}$), $\gamma_k = \text{rank}(\Gamma_{\rho_k})$

$M_{\rho_k} = \begin{bmatrix} 0 & I_{2n_y-\gamma_k} \end{bmatrix} U^T_{\rho_k}$

$\mathcal{F}_{\rho_k} = \mathcal{D}_{\rho_k} \mathcal{A}^+_{\rho_k}$, $\quad \mathcal{G}_{\rho_k} = M_{\rho_k}(I - \mathcal{A}_{\rho_k} \mathcal{A}^+_{\rho_k})$

$\mathcal{C}_{\rho_k} = \begin{bmatrix} C_{\rho_k} \\ C_{\rho_{k+1}} A_{\rho_k} \end{bmatrix}$, $\quad \mathcal{S}_{\rho_k} = \begin{bmatrix} 0_{n_y,n_v} \\ C_{\rho_{k+1}} F_{\rho_k} \end{bmatrix}$, $\quad \mathcal{V} = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix}$

$Z_k = ((A_{\rho_k} - \mathcal{F}_{\rho_k} \mathcal{C}_{\rho_k}) P_k \mathcal{C}^T_{\rho_k} \mathcal{G}^T_{\rho_k} + (F_{\rho_k} - \mathcal{F}_{\rho_k} \mathcal{S}_{\rho_k}) W \mathcal{S}^T_{\rho_k} \mathcal{G}^T_{\rho_k} - \mathcal{F}_{\rho_k} \mathcal{V} \mathcal{G}^T_{\rho_k})(\mathcal{G}_{\rho_k} \mathcal{C}_{\rho_k} P_k \mathcal{C}^T_{\rho_k} \mathcal{G}^T_{\rho_k} + \mathcal{G}_{\rho_k} \mathcal{S}_{\rho_k} W \mathcal{S}^T_{\rho_k} \mathcal{G}^T_{\rho_k} + \mathcal{G}_{\rho_k} \mathcal{V} \mathcal{G}^T_{\rho_k})^{-1}$

$L_{\rho_k} = \mathcal{F}_{\rho_k} + Z_k \mathcal{G}_{\rho_k}$

$Q_{\rho_k} = L_{\rho_k}(:, 1 : n_y) \qquad R_{\rho_k} = L_{\rho_k}(:, n_y + 1 : 2n_y$

$N_{\rho_k} = A_{\rho_k} - Q_{\rho_k} C_{\rho_k} - R_{\rho_k} C_{\rho_{k+1}} A_{\rho_k}$

$\hat{\mathbf{x}}_{k+1} = N_{\rho_k} \hat{\mathbf{x}}_k + Q_{\rho_k} \mathbf{y}_k + R_{\rho_k} \mathbf{y}_{k+1}$

$P_{k+1} = (A_{\rho_k} - L_{\rho_k} \mathcal{C}_{\rho_k}) P_k (A_{\rho_k} - L_{\rho_k} \mathcal{C}_{\rho_k})^T + (F_{\rho_k} - L_{\rho_k} \mathcal{S}_{\rho_k}) W (F_{\rho_k} - L_{\rho_k} \mathcal{S}_{\rho_k})^T + L_{\rho_k} \mathcal{V} L^T_{\rho_k}$

end for

example.
On the numerical point of view, the RMSE provided in table 2 confirm that the new observer clearly overpasses the others.

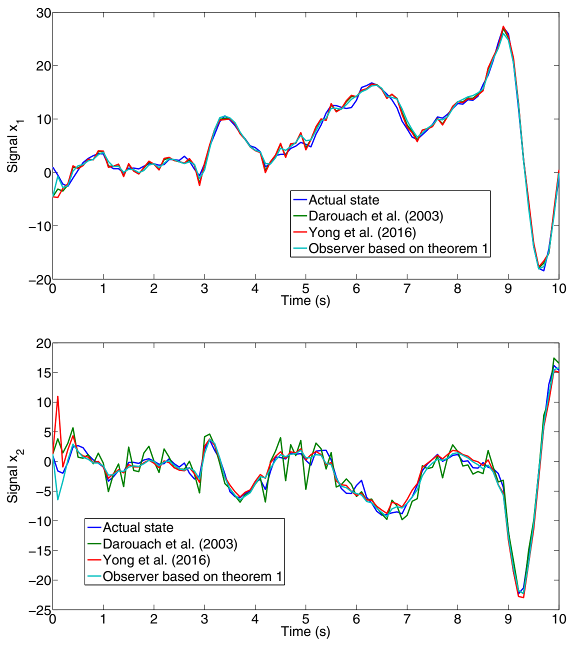

Fig. 2. Comparison of the estimations obtained by the observer of Darouach et al. (2003) and by the one of theorem 1.

Table 2. Mean Square Errors, in average on 100 simulations

Observer provided in	MSE on x_1	MSE on x_2
Darouach et al. (2003)	1.28	2.36
Yong et al. (2016)	1.32	1.68
Theorem 1	1.19	1.34

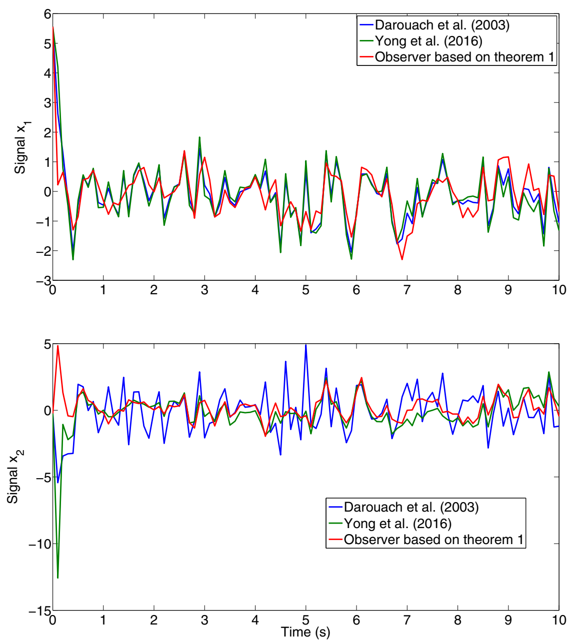

Fig. 3. Comparison of the estimation errors obtained by the observer of Darouach et al. (2003) and by the one of theorem 1.

4. EXTENSION

In this section, an important extension of theorem 1 will be seen. It is another observer that can be used in order to overpass the classical Linear Kalman Filter, in the case where the noise in the state equation has a very high variance. In that section, E_{ρ_k} is assumed to be equal to zero, that is the unknown input does not appear in the measurement equation (see remark 2).

4.1 Problem statement

In this section, we will deal with the following LPV structure:

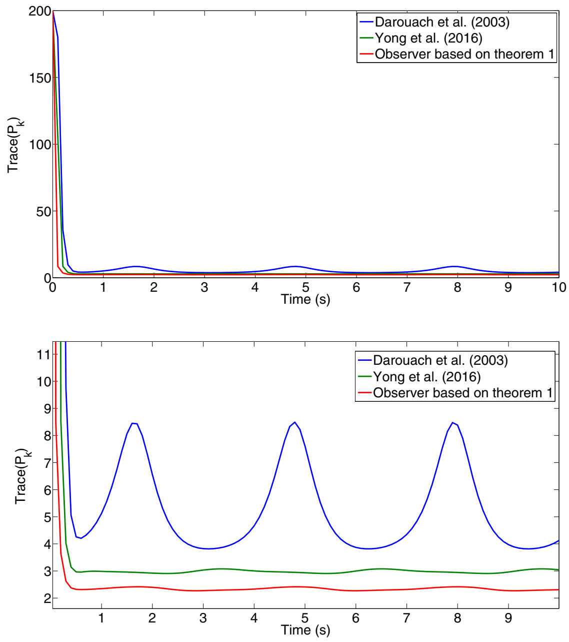

Fig. 4. Comparison of the trace of the covariance matrices obtained by the observer of Darouach et al. (2003) and by the observer of theorem 1 (the figure on the bottom is a zoom of the complete figure on the top).

$$\begin{cases} \mathbf{x}_{k+1} & = A_{\rho_k}\mathbf{x}_k + F_{\rho_k}\mathbf{v}_k \\ \mathbf{y_k} & = C_{\rho_k}\mathbf{x_k} + \mathbf{v}_k, \end{cases} \tag{20}$$

where $\mathbf{x}_k \in \mathbb{R}^{n_x}$ and $\mathbf{y}_k \in \mathbb{R}^{n_y}$ are the state and the output vector. and $\mathbf{w}_k \in \mathbb{R}^{n_v}$. , A_{ρ_k}, and F_{ρ_k} are two matrices parameter-varying with appropriate dimensions. C is a constant matrix. $\mathbf{v}_k \in \mathbb{R}^{n_y}$ is a zero-mean Gaussian white noise with covariance matrix V. $\mathbf{w}_k \in \mathbb{R}^{n_v}$ is *any noise*.

4.2 Main result

The second main result of the paper is presented in the following theorem.

Theorem 3. Under the following condition:

- $rank(CF_{\rho_k}) = rank(F_{\rho_k}), \forall \rho_k \in \Theta$,

the following equations provide an unbiased state estimator independent of the noise $\mathbf{w}_k$ and its characteristics, for the state estimation of the stochastic LPV system 20:

$$\begin{cases} \hat{\mathbf{x}}_{k+1} & = N_{\rho_k}\hat{\mathbf{x}}_k + Q_{\rho_k}\mathbf{y}_k + R_{\rho_k}\mathbf{y}_{k+1} \\ P_{k+1} & = (A_k - L_{\rho_k}\mathcal{C}_{\rho_k})P_k(A_k - L_{\rho_k}\mathcal{C}_{\rho_k})^T \\ & \quad + L_{\rho_k}\mathcal{V}L_{\rho_k}^T \\ N_{\rho_k} & = A_k - Q_{\rho_k}C_{\rho_k} - R_{\rho_k}C_{\rho_{k+1}}A_k \\ \left[Q_{\rho_k}\ \ R_{\rho_k}\right] & = L_{\rho_k} \\ L_{\rho_k} & = \mathcal{F}_{\rho_k} + Z_k\mathcal{G}_{\rho_k} \\ Z_k & = ((A_k - \mathcal{F}_{\rho_k}\mathcal{C}_{\rho_k})P_k\mathcal{C}_{\rho_k}^T\mathcal{G}_{\rho_k}^T \\ & \quad + (F_{\rho_k} - \mathcal{F}_{\rho_k}\mathcal{V}\mathcal{G}_{\rho_k}^T) \\ & \quad \times(\mathcal{G}_{\rho_k}\mathcal{C}_{\rho_k}P_k\mathcal{C}_{\rho_k}^T\mathcal{G}_{\rho_k}^T + \mathcal{G}_{\rho_k}\mathcal{V}\mathcal{G}_{\rho_k}^T)^{-1} \end{cases}, \tag{21}$$

where

$$\begin{aligned} \mathcal{F}_{\rho_k} &= F_{\rho_k}(C_{\rho_{k+1}}F_{\rho_k})^+, \\ \mathcal{G}_{\rho_k} &= M_{\rho_k}(I - (C_{\rho_{k+1}}F_{\rho_k})(C_{\rho_{k+1}}F_{\rho_k})^+), \\ \mathcal{C}_{\rho_k} &= \begin{bmatrix} C_{\rho_k} \\ C_{\rho_{k+1}}A_{\rho_k} \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix}, \end{aligned} \tag{22}$$

with $M_{\rho_k} = \left[0\ I_{2n_y - \gamma_k}\right] U_{\rho_k}^T$, where $C_{\rho_{k+1}}F_{\rho_k} = U_{\rho_k}\begin{bmatrix} \Gamma_{\rho_k} & 0 \\ 0 & 0 \end{bmatrix} T_{\rho_k}^T$ is the Singular Value Decomposition of $C_{\rho_{k+1}}F_{\rho_k}$ (Γ_{ρ_k} being a non singular matrix), and $\gamma_k = rank(C_{\rho_{k+1}}F_{\rho_k}) = rank(\Gamma_{\rho_k})$.

Proof. In order to prove this result, we only need to replace in theorem 1 D_{ρ_k} (resp. $\mathbf{d}_k$, F_{ρ_k}, W and $\mathbf{w}_k$) by F_{ρ_k} (resp. $\mathbf{w}_k$, 0, 0 and 0).

Remark 4. Two other cases can be discussed. The first one is another way of estimating the state of system (1). The idea is to write the state equation under the form:

$$\mathbf{x}_{k+1} = A_{\rho_k}\mathbf{x}_k + \overline{D}_{\rho_k}\overline{\mathbf{d}}_k \tag{23}$$

where $\overline{D}_{\rho_k} = [D_{\rho_k}\ \ F_{\rho_k}]$, and $\overline{\mathbf{d}}_k = \begin{bmatrix} \mathbf{d}_k \\ \mathbf{w}_k \end{bmatrix}$. Then we can apply the observer of theorem 1 if the rank condition $rank(C\overline{D}_{\rho_k}) = rank(\overline{D}_{\rho_k})$ is satisfied, which can also be written as:

$$rank([CD_{\rho_k}\ \ CF_{\rho_k}]) = rank([D_{\rho_k}\ \ F_{\rho_k}]). \tag{24}$$

If the previous condition is not satisfied, there is still a hope of decoupling a part of the noise $\mathbf{w}_k$. Indeed, let assume that $\mathbf{w}_k$ is a noise with a dimension strictly superior to 1, then it can be written under the form $\mathbf{w}_k = \begin{bmatrix} \mathbf{w}_{1,k} \\ \mathbf{w}_{2,k} \end{bmatrix}$, and thus F_{ρ_k} can be written $F_{\rho_k} = [F_{1,k}\ F_{2,k}]$, such that $F_{\rho_k}\mathbf{w}_k = [F_{1,k}\ F_{2,k}]\begin{bmatrix} \mathbf{w}_{1,k} \\ \mathbf{w}_{2,k} \end{bmatrix}$. If it exists such a decomposition of $\mathbf{w}_k$ that satisfies the following condition:

$$rank([CD_{\rho_k}\ \ CF_{1,k}]) = rank([D_{\rho_k}\ \ F_{1,k}]), \tag{25}$$

then the unknown input, as well as the noise $\mathbf{w}_{1,k}$ can be decoupled. The choice of the decomposition of $\mathbf{w}_k$ (if several are possible) may be done by considering the information available on each part $\mathbf{w}_{1,k}$ and $\mathbf{w}_{2,k}$ of the noise, or the importance of them variances.

4.3 Illustrative example

In order to illustrate the presented result, we will compare the proposed observer to the classical Linear Kalman Filter (LKF) on the following example.

$$\begin{cases} \mathbf{x}_{k+1} & = \begin{bmatrix} 1 & (1+\rho_k)dt \\ -\rho_k dt & 1-2dt \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 3 \end{bmatrix}\mathbf{w}_k \\ \mathbf{y_k} & = \left[1\ 0\right]\mathbf{x}_k + \mathbf{v}_k \end{cases}, \tag{26}$$

where $\mathbf{w}_k$ and $\mathbf{v}_k$ are zero-mean Gaussian white noises with covariance matrices $W = 10$ (we will also test the cases where $W = 1, W = 100, W = 1000$ and $W = 10000$), and $V = 1$ respectively. We take $\rho_k = 3cos(k.dt)^2 + 1$. We launch the simulation during 10 seconds with a step time $dt = 0.01s$.

It can be noted that the Linear Kalman Filter is totally designed to deal with that kind of system. Indeed, there is no unknown input, but only Gaussian white noises. However, we will see that the observer proposed in that section overpasses the Kalman Filter.

On the numerical point of view, when we have a look at the MSE provided in table 2, we can see that, despite of the increase of W, the MSE obtained with the observer we developed is stable, whereas the one obtained with the classical Kalman Filter increases with W, although, again, we only have Gaussian white noises in that example.

Thus we can conclude that, for state noise with high variance or with unknown characteristics (distribution, or expectancy or variance), the observer provided in theorem 3 is a very interesting alternative to the classical Linear Kalman Filter.

Table 3. Mean Square Errors, in average on 100 simulations (LKF : Linear Kalman Filter, UIKF: Unknown Input Kalman Filter)

MSE on	$W = 1$	$W = 10$	$W = 10^2$	$W = 10^3$	$W = 10^4$
LKF (x_1)	26.3	44.8	45.3	45.5	45.8
UIKF (x_1)	32.5	32.6	32.3	32.7	32.6
LKF (x_2)	75.8	237.8	315.9	334.9	363.4
UIKF (x_2)	99.5	100.2	98.9	99.2	99.5

5. CONCLUSIONS

In this paper, en observer dealing with unknown input in an LPV system with additive Gaussian white noise is proposed. This observer decouples the signal from the unknown input and is optimal (in a *minimum variance* sense). Numerical simulations show that it overpasses the observer designed in Darouach et al. (2003).

As an extension of this development, another observer is then constructed for LPV systems, and it is shown that it can be a good alternative to the Kalman filter in the cases of a state noise with very high variance, or if the characteristic of the state noise are not totally known.

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